Orthogonal Laurent Polynomials
of Jacobi, Hermite and Laguerre Types

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1 INTRODUCTION

We will outline in this paper a general procedure for constructing whole systems of
orthogonal Laurent polynomials on the real line from systems of orthogonal poly-
nomials. To further explain our intentions, we proceed with some basic definitions
and results, all of which appear, or are modifications of those that appear, in the
literature. In particular, [1,2,3,4,5,6,7,8,9,10,11,12] were consulted in our prepara-
tion.

If \( f: D \rightarrow \mathbb{R} \), where \( D \) is a subset of the set of real numbers \( \mathbb{R} \), then the set
\[
\sigma (f) := \{ x \in D : \text{There is an } \epsilon > 0 \text{ such that } (x - \epsilon, x + \epsilon) \subseteq D \text{ and } f(x + \delta) - f(x - \delta) > 0 \text{ for all } \delta > 0 \text{ such that } \delta < \epsilon \}
\]
is called the spectrum of \( f \). If \( \psi: \mathbb{R} \rightarrow \mathbb{R} \) is a bounded, non-decreasing function with
an infinite spectrum \( \sigma(\psi) \) such that the moments \( \mu_n(\psi) \) defined by the Riemann-
Stieltjes integrals
\[
\mu_n(\psi) := \int_{-\infty}^{\infty} x^n \, d\psi(x)
\]

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exist for all $n \in Z^0 := [0, 1, 2, ...]$, then $\psi$ is called a moment distribution function (MDF). If $\phi : (R^- \cup R^+) \to R$ is a bounded function, non-decreasing on the negative reals $R^-$ and the positive reals $R^+$ separately, with infinite spectrum $\sigma(\phi)$ such that the moments
\[
\mu_n(\phi) := \int_{-\infty}^{0} x^n \, d\phi(x) + \int_{0}^{\infty} x^n \, d\phi(x)
\]
exist for all $n \in Z := [0, \pm 1, \pm 2, ...]$, then $\phi$ is called a strong moment distribution function (SMDF).

We denote by $P$ the space of all polynomials in $x$ with real coefficients, and we denote by $R$ the space of all Laurent polynomials (L-polynomials), $\sum_{k=m}^{n} r_k x^k$ for $m, n \in Z$ with $m \leq n$, having real coefficients $r_k$ for $m \leq k \leq n$. For $R \in R$ we sometimes write $C_k(R)$ for the coefficient of $x^k$ in $R$. We will find it useful to define the following subsets of $R$:
\[
R_{m,n} := \left\{ \sum_{k=m}^{n} r_k x^k : r_k \in R, \ m \leq k \leq n \right\} \text{ for } m, n \in Z \text{ with } m \leq n;
\]
\[
R_{2n} := \left\{ R \in R_{-n,n} : C_n(R) \neq 0 \right\} \text{ for } n \in Z_0^+;
\]
\[
R_{2n+1} := \left\{ R \in R_{-n-1,n} : C_{-n-1}(R) \neq 0 \right\} \text{ for } n \in Z_0^+.
\]

THEOREM 1.1 For any non-zero $R \in R$, there is a unique $d \in Z_0^+$, called the $L$-degree of $R$, such that $R \in R_d$.
Proof: See [4], p. 48, Theorem 1.1.

Notice that for a polynomial in $P$, its L-degree is twice its polynomial degree. If $R \in R_{2n}$, then $C_n(R)$ and $C_{-n}(R)$ are called the leading coefficient and trailing coefficient of $R$, respectively, and if $R \in R_{2n+1}$, then the leading coefficient and trailing coefficient of $R$ are $C_{-n-1}(R)$ and $C_n(R)$, respectively. An L-polynomial is called monic if its leading coefficient is 1, regular if its trailing coefficient is non-zero, and singular if its trailing coefficient is 0.

For any MDF $\psi$ and polynomials $P, Q \in P$, we define
\[
(P, Q)_\psi := \int_{-\infty}^{\infty} P(x)Q(x) \, d\psi(x),
\]
and, for any SMDF $\phi$ and L-polynomials $R, S \in R$, we set
\[
(R, S)_\phi := \int_{-\infty}^{0} R(x)S(x) \, d\phi(x) + \int_{0}^{\infty} R(x)S(x) \, d\phi(x).
\]

THEOREM 1.2 If $\psi$ is an MDF, then $(\cdot, \cdot)_\psi$ is an inner-product on $P$.
Proof: See [3], p.13 and p.16.

THEOREM 1.3 If $\phi$ is an SMDF, then $(\cdot, \cdot)_\phi$ is an inner-product on $R$. 

Proof: See [4], p.58 and p.62, Theorem 3.6.

A sequence $\{P_n(x)\}_{n=0}^\infty$ in $P$ such that, for $\psi$, an MDF, and all $m, n \in \mathbb{Z}_0^+$,

$$P_n \text{ has polynomial degree } n \text{ and } (P_m, P_n)_\psi = 0 \text{ if } m \neq n,$$

is called an orthogonal polynomial sequence (OPS) with respect to $\psi$. If in addition each of the polynomials $P_n$ is monic, $\{P_n(x)\}_{n=0}^\infty$ is called monic. It is not difficult to show that a monic OPS with respect to a given MDF $\psi$ exists and is unique (see [4], Theorem 3.3, p.14, and Corollary, p.9). Analogously in the Laurent case, a sequence $\{R_n(x)\}_{n=0}^\infty$ in $R$ is called an orthogonal Laurent polynomial sequence (OLPS) with respect to $\phi$, an SMDF, if

$$R_n \text{ has } L\text{-degree } n \text{ and } (R_m, R_n)_\phi = 0 \text{ if } m \neq n.$$

$\{R_n(x)\}_{n=0}^\infty$ is called monic if each of the L-polynomials $R_n$ is monic. Again, it is not hard to show that a unique monic OLPS with respect to a given SMDF $\phi$ exists (see [4], Theorem 3.2, p.59, and Corollary 1.6, p.53).

Our main goal in the next section is to show that, given positive real parameters $\gamma$ and $\lambda$, the L-polynomial transformation $x \to 1/\lambda(x - \gamma/x)$ takes systems of OPS’s into systems of OLPS’s. This and other results of the preceding sections are preparations for our exposition of orthogonal L-polynomials of Jacobi, Hermite and Laguerre types, contained in the third, and final, part of this paper.

2 THE TRANSFORMATION

In this section, we present a rigorous exposition of a general, parameterized L-polynomial transformation of OPS’s into OLPS’s. Although our work here is of a general nature, we anticipate specific applications, presented in Section 3, to Jacobi, Hermite and Laguerre systems of OPS’s.

2.1 Definitions

For the rest of Section 2, we assume that

$\psi$ is an MDF with monic OPS $\{P_n(x)\}_{n=0}^\infty$

and that

$\lambda$ and $\gamma$ are fixed positive real numbers.

Our development will be facilitated by a series of definitions: We set

$$v(x) := \frac{1}{\lambda} \left( x - \frac{\gamma}{x} \right),$$

$$v^{-1}_\pm(y) := \frac{\lambda}{2} \left( y \pm \sqrt{y^2 - \frac{4\gamma}{\lambda^2}} \right),$$
\[ \tilde{\psi}(x) := \begin{cases} \int_{-\infty}^{x} \frac{1}{v(t)} \, d(\psi \circ v)(t), & \text{for } x \in \mathbb{R}^- \\ \int_{0}^{x} \frac{1}{v(t)} \, d(\psi \circ v)(t), & \text{for } x \in \mathbb{R}^+ \end{cases} \]

and, for \( n = 0, 1, 2, \ldots \),

\[ \tilde{P}_{2n}(x) := \lambda^n P_n(v(x)) \text{ and } \tilde{P}_{2n+1}(x) := \left(-\frac{\lambda}{\gamma}\right)^n \frac{1}{x} P_n(v(x)). \]

### 2.2 Preliminary Theorems

We begin with an exploration of the essential component, \( v \), and the related functions, \( v_-^{-1}, v_+^{-1} \), and \( v' \), of our transformation.

**THEOREM 2.2.1**

(A) \( v|_{\mathbb{R}^-}, v|_{\mathbb{R}^+}, v_-^{-1}, \) and \( v_+^{-1} \) are differentiable, monotone increasing functions.

(B) \( v|_{\mathbb{R}^+} \) is a diffeomorphism from \( \mathbb{R}^+ \) to \( \mathbb{R} \). Its inverse is \( v_+^{-1} \).

(C) \( v|_{\mathbb{R}^-} \) is a diffeomorphism from \( \mathbb{R}^- \) to \( \mathbb{R} \). Its inverse is \( v_-^{-1} \).

(D) For all \( x \in \mathbb{R}^- \cup \mathbb{R}^+ \), \( v(-\frac{2}{x}) = v(x) \).

(E) For all \( t \in \mathbb{R}^- \cup \mathbb{R}^+ \), \( \frac{dv}{dx} \bigg|_{x=\frac{1}{t}} = \frac{t^2 \, dv}{dx} \bigg|_{x=t} \).

**Proof:**

(A) Evidently, \( v \) has domain \( \mathbb{R}^- \cup \mathbb{R}^+ \), and \( \frac{dv}{dx} = \frac{1}{\lambda}(1 + \frac{\gamma}{2x}) > \frac{1}{\lambda} > 0 \); that is, \( v \) is differentiable and monotone increasing on \( \mathbb{R}^- \) and on \( \mathbb{R}^+ \) by elementary calculus techniques. Similarly, it is clear that the domain of \( v_+^{-1} \) is \( \mathbb{R} \), and, since

\[ \left| y/\sqrt{y^2 + \frac{4\gamma}{\lambda^2}} \right| < 1, \quad \frac{v_+^{-1}}{dy} = \frac{\lambda}{2} \left(1 \pm y/\sqrt{y^2 + \frac{4\gamma}{\lambda^2}}\right) > 0; \text{ that is, again by elementary calculus, } v_+^{-1} \text{ is differentiable and monotone increasing.} \]

(B) By inspection of the definitions, \( v|_{\mathbb{R}^+} \) maps \( \mathbb{R}^+ \) onto \( \mathbb{R} \), and \( v_+^{-1} \) maps \( \mathbb{R} \) onto \( \mathbb{R}^+ \). By Theorem 2.2.1 (A), \( v|_{\mathbb{R}^+} \) and \( v_+^{-1} \) are injective and differentiable. Hence, it suffices to show that \( v_+^{-1}(v(x)) = x \), for any \( x \) in \( \mathbb{R}^+ \). But, for \( x \) in \( \mathbb{R}^+ \),

\[ v_+^{-1}(v(x)) = \frac{\lambda}{2} \left(1 \pm \frac{\sqrt{\frac{1}{\lambda}(x - \frac{\gamma}{x})^2 + \frac{4\gamma}{\lambda^2}}}{\sqrt{\frac{1}{\lambda}(x - \frac{\gamma}{x})^2 + \frac{4\gamma}{\lambda}}}\right) \]

\[ = \frac{\lambda}{2} \left(1 \pm \frac{\sqrt{\frac{1}{\lambda}(x^2 + 2\gamma + \frac{\gamma^2}{x^2})}}{\sqrt{\frac{1}{\lambda}(x^2 + 2\gamma + \frac{\gamma^2}{x^2})}}\right) \]

\[ = \frac{\lambda}{2} \left(1 \pm \frac{\sqrt{\frac{1}{\lambda}(x + \frac{\gamma}{x})^2}}{\sqrt{\frac{1}{\lambda}(x + \frac{\gamma}{x})^2}}\right) \]

\[ = \frac{\lambda}{2} \left(1 \pm \frac{\sqrt{\frac{1}{\lambda}(x + \frac{\gamma}{x})^2}}{\sqrt{\frac{1}{\lambda}(x + \frac{\gamma}{x})^2}}\right) \]

\[ = x. \]

(C) A proof can be given in exactly the same way as that for Theorem 2.2.1 (B).

(D) \( v(x) := \frac{1}{\lambda}(x - \frac{\gamma}{x}) = \frac{1}{\lambda}(\frac{\gamma}{x} - \gamma/(\frac{\gamma}{x})) = v(-\frac{2}{x}), \text{ for any non-zero } x \in \mathbb{R} \).
Proof: Suppose \( \frac{d}{dx} |_{x=\tilde{\varphi}} = \frac{1}{x} (1 + \frac{\gamma}{x^2}) \) for any non-zero \( t \) in \( \mathbb{R} \).

**THEOREM 2.2.2**

(A) \( \tilde{\varphi} \) is a bounded function on \( \mathbb{R}^+ \cup \mathbb{R}^- \).

(B) \( \tilde{\varphi} \) is non-decreasing on \( \mathbb{R}^- \) and \( \mathbb{R}^+ \) separately.

Proof: (A) Inspection of the definitions shows that \( 0 < \frac{1}{v'(t)} < \lambda \), for all non-zero \( t \) in \( \mathbb{R} \). Hence, by comparison, for \( x \) in \( \mathbb{R}^- \),

\[
0 \leq \int_{-\infty}^{\psi(x)} \frac{1}{v'(y)} d\psi(y) \leq \int_{-\infty}^{\psi(x)} \lambda d\psi(y) = \lambda \mu_0(\psi).
\]

But, for \( x \) in \( \mathbb{R}^- \), with \( y = v(t) \),

\[
\tilde{\psi}(x) := \int_{-\infty}^{x} \frac{1}{v'(t)} d(\psi \circ v)(t) = \int_{-\infty}^{\psi(x)} \frac{1}{v'(y)} d\psi(y).
\]

Hence, \( 0 \leq \tilde{\psi}(x) \leq \lambda \mu_0(\psi) \), for \( x \) in \( \mathbb{R}^- \); that is, \( \tilde{\psi} \) is a bounded map from \( \mathbb{R}^- \) to \( \mathbb{R} \). A similar argument shows that \( \tilde{\psi} \) is a bounded map from \( \mathbb{R}^+ \) to \( \mathbb{R} \).

(B) To verify that \( \tilde{\psi} \) is non-decreasing on \( \mathbb{R}^- \), suppose \( -\infty < x < y < 0 \). For all \( t \leq y \), \( \frac{1}{v'(t)} \geq \frac{1}{v'(y)} \). Hence, \( \psi(y) - \psi(x) = \int_{x}^{y} \frac{1}{v'(t)} d(\psi \circ v)(t) \geq \int_{x}^{y} \frac{1}{v'(y)} d(\psi \circ v)(t) = \frac{\psi(y) - \psi(x)}{v'(y)} \).

But, \( v'(y) = \frac{1}{x} (1 + \frac{\gamma}{x^2}) > 0 \), and \( \psi(v(y)) = \psi(v(x)) \geq 0 \) by the monotonicity of \( \psi \) and \( v \). Thus, \( \tilde{\psi}(y) - \tilde{\psi}(x) \geq 0 \); that is, \( \tilde{\psi} \) is non-decreasing on \( \mathbb{R}^- \). Likewise, it can be shown that \( \tilde{\psi} \) is non-decreasing on \( \mathbb{R}^+ \).

**THEOREM 2.2.3** \( \sigma(\tilde{\psi}) = v_{-1}^{-1}(\sigma(\psi)) \cup v_{+1}^{-1}(\sigma(\psi)) \).

Proof: Suppose \( x \in v_{-1}^{-1}(\sigma(\psi)) \). Then \( x \in \mathbb{R}^- \), and \( v(x) = v(v_{-1}(\sigma(\psi))) = \sigma(\psi) \). Therefore, there is an \( \epsilon > 0 \) such that \( (x-\epsilon, x+\epsilon) \subset \mathbb{R}^- \), and \( \psi(v(x)+\delta) - \psi(v(x)-\delta) > 0 \) for all \( \delta > 0 \). Let \( \delta_1 \) satisfy \( 0 < \delta_1 < \epsilon \). We have \( v(x-\delta_1) < v(x) < v(x+\delta_1) \) by Theorem 2.2.1 (A). Hence, there exists a \( \delta_2 > 0 \) such that \( v(x-\delta_1) < v(x) - \delta_2 < v(x) + \delta_2 < v(x+\delta_1) \), and the estimates

\[
\tilde{\psi}(x + \delta_1) - \tilde{\psi}(x - \delta_1) = \int_{v(x-\delta_1)}^{v(x+\delta_1)} \frac{1}{v'(v_{-1}(y))} d\psi(y)
\]

\[
= \frac{1}{v'(x+\delta_1)} \int_{v(x-\delta_1)}^{v(x+\delta_1)} d\psi(y)
\]

\[
= \frac{1}{v'(x+\delta_1)} (\psi(v(x+\delta_1)) - \psi(v(x-\delta_1)))
\]

hold. But \( \frac{1}{v'(x+\delta_1)} > 0 \), and, since \( v(x) \) is in \( \sigma(\psi) \), \( \psi(v(x)+\delta_2) - \psi(v(x)-\delta_2) > 0 \).

Thus, \( \tilde{\psi}(x + \delta_1) - \tilde{\psi}(x - \delta_1) > 0 \). It follows that \( v_{-1}(\sigma(\psi)) \subseteq \sigma(\tilde{\psi}) \). A similar argument shows that \( v_{+1}^{-1}(\sigma(\psi)) \subseteq \sigma(\tilde{\psi}) \). Thus, \( (v_{-1}^{-1}(\sigma(\psi)) \cup v_{+1}^{-1}(\sigma(\psi)) \subseteq \sigma(\tilde{\psi}) \).
Next, suppose \( x \in (\sigma(\tilde{\psi}) \cap \mathbf{R}^-) \). By Theorem 2.2.1 (C), there is a unique \( y \in \mathbf{R} \) such that \( v_-(y) = x \). But estimates similar to those above show that \( y \in \sigma(\psi) \). Thus, \( x \in v_-(\sigma(\psi)) \) if \( x \in (\sigma(\tilde{\psi}) \cap \mathbf{R}^-) \), and hence \( (\sigma(\tilde{\psi}) \cap \mathbf{R}^-) \subseteq v_-(\sigma(\psi)) \). In an analogous manner, it can be shown that \( (\sigma(\tilde{\psi}) \cap \mathbf{R}^+) \subseteq v_+(\sigma(\psi)) \). Thus, it follows that \( \sigma(\tilde{\psi}) = \left((\sigma(\tilde{\psi}) \cap \mathbf{R}^-) \cup (\sigma(\tilde{\psi}) \cap \mathbf{R}^+)\right) \subseteq \left(v_-(\sigma(\psi)) \cup v_+(\sigma(\psi))\right) \).

**THEOREM 2.2.4** Let \( n \) be any integer. Then:

(A) \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) \) and \( \int_{0}^{\infty} x^n v'(x) d(\psi \circ v)(x) \) exist.

(B) \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} v'(x) d(\psi \circ v)(x) \).

(C) \( \int_{-\infty}^{0} x^n d\tilde{\psi}(x) = \int_{0}^{\infty} x^n v'(x) d(\psi \circ v)(x) \) and \( \int_{-\infty}^{0} x^n d\tilde{\psi}(x) = \int_{0}^{\infty} x^n v'(x) d(\psi \circ v)(x) \).

(D) \( \int_{-\infty}^{0} x^n d\tilde{\psi}(x) \) and \( \int_{0}^{\infty} x^n d\tilde{\psi}(x) \) exist.

(E) \( \int_{-\infty}^{0} x^n d\tilde{\psi}(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} d\tilde{\psi}(x) \).

**Proof:** (A) Since \( 0 < \frac{1}{v'(x)} < \lambda \), and, for all \( x \in \mathbf{R}^- \),

\[
0 \leq |x| = \left| v_-(v(x)) \right| = \frac{\lambda}{2} \left( v(x) - \sqrt{v^2(x) + \frac{4\sigma}{\lambda^2}} \right) \leq \frac{\lambda}{2} \left( |v(x)| + |v(x)| + \frac{2\sqrt{\gamma}}{\lambda} \right) \leq \lambda |v(x)| + \sqrt{\gamma},
\]

we have \( 0 \leq |x|^n v'(x) \leq \lambda (|v(x)| + \sqrt{\gamma})^n \) for all \( x \in \mathbf{R}^- \) and \( n \in \mathbf{Z}^+_0 \). Since the moments \( \mu_n(\psi) \) are finite for \( n \in \mathbf{Z}_0^+ \), we can deduce that the integrals \( \int_{-\infty}^{\infty} |t|^n d\psi(t) \) are finite by comparing \( |t|^n \) to \( t^{n+1} \), for \( N \) an even integer greater than \( n \). Hence, for \( n \in \mathbf{Z}^+_0 \), the integrals \( \int_{-\infty}^{\infty} \lambda |t|^n + \sqrt{\gamma} d\psi(t) = \int_{0}^{\infty} \lambda |v(x)| + \sqrt{\gamma} d(\psi \circ v)(x) \) exist, and the integrals \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) \) exist, by comparison. A similar argument shows that the integrals \( \int_{0}^{\infty} x^n v'(x) d(\psi \circ v)(x) \), for \( n \in \mathbf{Z}^+_0 \), exist.

The substitution \( x \to -\frac{2}{x} \) in \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) \) yields, by Theorem 2.2.1, parts (D) and (E),

\[
\int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} v'(x) d(\psi \circ v)(x).
\]

Hence, the integrals \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) \) and \( \int_{0}^{\infty} x^n v'(x) d(\psi \circ v)(x) \) exist for all integers \( n \), with the possible exception of the case \( n = -1 \). But, a comparison of \( |x|^{-1} \) to \( x^{-2} + 1 \) now shows that the integrals exist also for \( n = -1 \).

(B) By Theorem 2.2.4 (A), the integral \( \int_{-\infty}^{0} x^n v'(x) d(\psi \circ v)(x) \) exists for any integer \( n \). As in the proof of Theorem 2.2.4 (A), the substitution \( x \to -\frac{2}{x} \) now
yields
\[
\int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} \frac{1}{v'(x)} d(\psi \circ v)(x),
\]
for any integer \( n \).

(C, D) Let \( n \) be an integer, and suppose \( \infty < a < b < 0 \). Then, since the integrands are continuous and the integrators are non-decreasing and bounded on the closed interval \([a, b]\), the integrals
\[
\int_{a}^{b} x^n d\tilde{\psi}(x) \quad \text{and} \quad \int_{a}^{b} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)
\]
exist for all \( n \in \mathbb{Z} \).

Next, set \( x_{m,k} := k \frac{b-a}{m} + a \) for \( m \geq 1 \) and \( k = 0, 1, 2, \ldots, m \). By the Mean Value Theorem, there is a \( c_{m,k} \) in the closed interval \([x_{m,k-1}, x_{m,k}]\) such that
\[
\int_{x_{m,k}}^{x_{m,k-1}} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1})))
\]
for each \( k = 1, 2, \ldots, m \). Since the integrals exist, we can choose to take
\[
\int_{a}^{b} x^n d\tilde{\psi}(x) = \lim_{m \to \infty} \sum_{k=1}^{m} c_{m,k}^{n} (\tilde{\psi}(x_{m,k}) - \tilde{\psi}(x_{m,k-1}))
\]
and
\[
\int_{a}^{b} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = \lim_{m \to \infty} \sum_{k=1}^{m} c_{m,k}^{n} \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1}))).
\]

But, the definition of \( \tilde{\psi} \) and additivity of the integral imply
\[
\tilde{\psi}(x_{m,k}) - \tilde{\psi}(x_{m,k-1}) = \int_{x_{m,k-1}}^{x_{m,k}} \frac{1}{v'(x)} d(\psi \circ v)(x).
\]

It follows that
\[
\int_{a}^{b} x^n d\tilde{\psi}(x) = \lim_{m \to \infty} \sum_{k=1}^{m} c_{m,k}^{n} \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1})))
\]
\[
= \int_{a}^{b} x^n \frac{1}{v'(x)} d(\psi \circ v)(x).
\]

Since \( \int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \) exists by Theorem 2.2.4 (A), we then have \( \int_{-\infty}^{0} x^n d\tilde{\psi}(x) \) exists and equals \( \int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \).

Likewise, it follows that \( \int_{0}^{\infty} x^n d\tilde{\psi}(x) \) exists and is equal to \( \int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \).

(E) The result follows by substitution of the integrals in Theorem 2.2.4 (C) in the
equation of Theorem 2.2.4 (B).

THEOREM 2.2.5 Let $n$ be any integer. Then:

(A) $\mu_n(\tilde{\psi})$ exists.

(B) $\mu_n(\tilde{\psi}) = (-1)^n \gamma^{n+1} \mu_{-n-2}(\tilde{\psi})$.

(C) $\mu_{-1}(\tilde{\psi}) = 0$.

Proof: (A) Since $\mu_n(\tilde{\psi}) := \int_{-\infty}^{0} x^n d\tilde{\psi}(x) + \int_{0}^{\infty} x^n d\tilde{\psi}(x)$ by definition, Theorem 2.2.4 (D) implies $\mu_n(\tilde{\psi})$ exists for any integer $n$.

(B) The result follows immediately by applying Theorem 2.2.4 (E) to the definition $\mu_n(\tilde{\psi}) := \int_{-\infty}^{0} x^n d\tilde{\psi}(x) + \int_{0}^{\infty} x^n d\tilde{\psi}(x)$.

(C) By Theorem 2.2.5 (A), $\mu_{-1}(\tilde{\psi})$ exists, and, by Theorem 2.2.5 (B) with $n = -1$, $\mu_{-1}(\tilde{\psi}) = -\mu_{-1}(\tilde{\psi})$. Hence, $\mu_{-1}(\tilde{\psi}) = 0$.

THEOREM 2.2.6 $\tilde{\psi}$ is a SMDF.

Proof: By Theorem 2.2.2, $\tilde{\psi}$ is a bounded function, non-decreasing on $\mathbb{R}^-$ and $\mathbb{R}^+$ separately. Theorem 2.2.1 (C) implies $v^{-1}$ and $v_+^{-1}$ are one-to-one, and Theorem 2.2.3 says $\sigma(\tilde{\psi}) = v^{-1}(\sigma(\psi)) \cup v_+^{-1}(\sigma(\psi))$. Hence, $\sigma(\tilde{\psi})$ is infinite since $\sigma(\psi)$ is infinite, $\tilde{\psi}$ being a MDF. Lastly, the moments $\mu_n(\tilde{\psi})$, for each integer $n$, exists by Theorem 2.2.5 (A).

THEOREM 2.2.7 Let $R$ and $S$ be Laurent polynomials. Then the inner-product $(R, S)_{\tilde{\psi}} = \int_{-\infty}^{0} R(x)S(x) \frac{1}{\nu(x)} d(\psi \circ \nu)(x) + \int_{0}^{\infty} R(x)S(x) \frac{1}{\nu(x)} d(\psi \circ \nu)(x)$.

Proof: By Theorem 2.2.6, $\tilde{\psi}$ is a SMDF. Hence, by definition,

$$ (R, S)_{\tilde{\psi}} = \int_{-\infty}^{0} R(x)S(x) \frac{1}{\nu(x)} d\tilde{\psi}(x) + \int_{0}^{\infty} R(x)S(x) \frac{1}{\nu(x)} d\tilde{\psi}(x). $$

Thus, the result follows from Theorem 2.2.4 (C) and linearity of the integral.

THEOREM 2.2.8 Let $j$ and $k$ be non-negative integers. Then:

(A) $(\tilde{P}_{2j}, \tilde{P}_{2k})_{\tilde{\psi}} = \lambda^{j+k+1} (P_{2j}, P_{2k})$. 

(B) $(\tilde{P}_{2j+1}, \tilde{P}_{2k+1})_{\tilde{\psi}} = \left( \frac{1}{\gamma} \right)^{j+k+1} (P_{2j}, P_{2k})$. 

(C) $(\tilde{P}_{2j+1}, \tilde{P}_{2k})_{\tilde{\psi}} = 0$.

Proof: (A) The definition of $(\tilde{P}_{2j}, \tilde{P}_{2k})_{\tilde{\psi}}$, Theorem 2.2.7, the substitution $x = -\frac{t}{\gamma}$, Theorem 2.2.1 (E), linearity of the integral, the definition of $P_{2j}(x)$ and $P_{2k}(x)$, the
substitution \( y = v(x) \), and the definition of \((P_j, P_k)_\psi\) justify

\[
(\bar{P}_{2j}, \bar{P}_{2k})_\psi := \int_{-\infty}^{0} \bar{P}_{2j}(x)\bar{P}_{2k}(x) d\tilde{\psi}(x) + \int_{0}^{\infty} \bar{P}_{2j}(x)\bar{P}_{2k}(x) d\tilde{\psi}(x)
= \int_{-\infty}^{0} \bar{P}_{2j}(x)\bar{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \\
\int_{0}^{\infty} \bar{P}_{2j}(x)\bar{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x)
= \int_{0}^{\infty} \bar{P}_{2j}(t)\bar{P}_{2k}(t) \frac{\gamma}{t^2} \frac{1}{v'(t)} d(\psi \circ v)(t) + \\
\int_{0}^{\infty} \bar{P}_{2j}(x)\bar{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x)
= \lambda \int_{0}^{\infty} \bar{P}_{2j}(x)\bar{P}_{2k}(x) \frac{1}{\lambda} \left(1 + \frac{\gamma}{x^2}\right) \frac{1}{v'(x)} d(\psi \circ v)(x)
= \lambda \int_{0}^{\infty} \bar{P}_{2j}(x)\bar{P}_{2k}(x) d(\psi \circ v)(x)
= \lambda^{j+k+1} \int_{0}^{\infty} P_j(v(x))P_k(v(x)) d(\psi \circ v)(x)
= \lambda^{j+k+1} \int_{-\infty}^{\infty} P_j(y)P_k(y) d\psi(y)
= \lambda^{j+k+1}(P_j, P_k)_\psi.
\]

(B) By similar means as used in the proof of Theorem 2.2.8 (A),

\[
(\bar{P}_{2j+1}, \bar{P}_{2k+1})_\psi = (-1)^{j+k} \left(\frac{\lambda}{\gamma}\right)^{j+k+1} (P_j, P_k)_\psi.
\]

If \( j \neq k \), then \((P_j, P_k)_\psi = 0\) by orthogonality. If \( j = k \), then \((-1)^{j+k} = 1\). In either case,

\[
(-1)^{j+k} \left(\frac{\lambda}{\gamma}\right)^{j+k+1} (P_j, P_k)_\psi = \left(\frac{\lambda}{\gamma}\right)^{j+k+1} (P_j, P_k)_\psi.
\]

The result therefore follows.

(C) By arguments similar to those used in the proofs of the previous two parts of Theorem 2.2.8,

\[
(\bar{P}_{2j+1}, \bar{P}_{2k})_\psi = -(\bar{P}_{2j+1}, \bar{P}_{2k})_\psi.
\]

Then, since \((\bar{P}_{2j+1}, \bar{P}_{2k})_\psi\) is finite by Theorem 2.2.6, we must have \((\bar{P}_{2j+1}, \bar{P}_{2k})_\psi = 0\).

THEOREM 2.2.9 \{\bar{P}_n(x)\}_{n=0}^\infty is the monic OLPS with respect to \(\tilde{\psi}\).

Proof: Inspection of the definition of \(\bar{P}_n(x)\) shows that it is a monic L-polynomial of L-degree \(n\), and Theorem 2.2.8 implies orthogonality of \{\bar{P}_n(x)\}_{n=0}^\infty with respect to \(\tilde{\psi}\).
2.3 The Transformation Theorem

For ease of reference and discussion we collect several of the results obtained in the previous section into the following theorem.

THEOREM 2.3.1 (The Transformation Theorem) Let $\psi$ be a moment distribution function, let $\sigma(\psi)$ denote the spectrum of $\psi$, and let $\{P_n(x)\}_{n=0}^{\infty}$ denote the monic orthogonal polynomial sequence with respect to $\psi$. Let $\lambda, \gamma \in \mathbb{R}^*$, and set $v(x) := \frac{1}{x} \left( x - \frac{\gamma}{x} \right)$ and $v_\pm^{-1}(y) := \frac{1}{2} \left( y \pm \sqrt{y^2 + \frac{4 \gamma^2}{x^2}} \right)$. Then:

(A) $\tilde{\psi}(x) := \begin{cases} \int_{-\infty}^{x} \frac{1}{v(t)} \, d(\psi \circ v)(t), & x \in \mathbb{R}^- \quad \text{is a strong moment distribution function.} \\ \int_{x}^{\infty} \frac{1}{v(t)} \, d(\psi \circ v)(t), & x \in \mathbb{R}^+ \end{cases}$

(B) $\sigma(\tilde{\psi}) = v^{-1}(\sigma(\psi)) \cup v_+^{-1}(\sigma(\psi))$ is the spectrum of $\tilde{\psi}$.

(C) $\{\tilde{P}_m(x)\}_{m=0}^{\infty}$, where $\tilde{P}_n(x) := \lambda^n P_n(v(x))$ and $\tilde{P}_{2n+1}(x) := \lambda^n P_n(v(x))$ for $n = 0, 1, 2, \ldots$, is the monic orthogonal Laurent polynomial sequence with respect to $\tilde{\psi}$.

Proof: See the proofs of Theorem 2.2.3, Theorem 2.2.6 and Theorem 2.2.9.

We call $v$ the doubling transformation because it is a monotone increasing function of both $\mathbb{R}^-$ and $\mathbb{R}^+$ onto $\mathbb{R}$. In effect, $(f \circ v)|_{\mathbb{R}^-}$ and $(f \circ v)|_{\mathbb{R}^+}$ are copies of $f : \mathbb{R} \to \mathbb{R}$ living on the negative reals and the positive reals, respectively. In this sense, $f \circ v$ is a doubling of $f$. Of course, $v$ is not the only monotone increasing function of both $\mathbb{R}^-$ and $\mathbb{R}^+$ onto $\mathbb{R}$; that is, $v$ is not the only doubling transformation. However, $v(x) := \frac{1}{x} \left( x - \frac{\gamma}{x} \right)$ is a Laurent polynomial. This feature, along with those given in Theorem 2.2.1, make $v$ especially useful for the purpose of transforming systems of OPS’s into systems of OLPS’s. Inspection of the Transformation Theorem shows that the L-polynomial $\tilde{P}_{2n}$ is a doubling of the polynomial $P_n$, and, in a slightly looser sense, $\psi$ with spectrum $\sigma(\psi) = v^{-1}(\sigma(\psi)) \cup v_+^{-1}(\sigma(\psi))$ is a doubling of $\psi$ with spectrum $\sigma(\tilde{\psi})$.

The doubling of the spectrum in particular can be used to discuss to what extent $\tilde{\psi}$ is an extension of $\psi$. For example, if $\sigma(\psi)$ is a symmetric set about the origin, it can be seen by Theorem 2.3.1 (B) and the definitions of $v^{-1}$ and $v_+^{-1}$ that $\sigma(\tilde{\psi})$ is symmetric about the origin. When $\sigma(\psi)$ is a symmetric interval about the origin, $\sigma(\tilde{\psi})$ is the union of two disjoint intervals forming a set symmetric about the origin. In particular, if $\sigma(\psi) = \mathbb{R}$, then $\sigma(\tilde{\psi}) = \mathbb{R}^- \cup \mathbb{R}^+$. If $\sigma(\psi) \subseteq \mathbb{R}^*_0 = [0, \infty)$, we would like an extension of $\psi$ to a SMDF to have its spectrum contained in $\mathbb{R}^+$. However, a direct application of the Transformation Theorem to a MDF $\psi$ having spectrum $\sigma(\psi) \subseteq \mathbb{R}^*_0$ yields the SMDF $\psi$ with its doubled spectrum $\sigma(\psi) = v^{-1}(\sigma(\psi)) \cup v_+^{-1}(\sigma(\psi))$ half contained in $\mathbb{R}^-$. Similarly, if $\sigma(\psi) \subseteq \mathbb{R}^*_0 = (-\infty, 0]$, then $\sigma(\tilde{\psi})$ is half contained in $\mathbb{R}^-$.

In a further effort to discover to what extent the transformed objects given by the Transformation Theorem are extensions of the corresponding original objects, it is worth examining the limiting case of $\lambda = 1$ and $\gamma = 0$. In this situation, it can be seen by inspection of the definitions that $v(x) = x$ and $v_\pm^{-1}(x) = x I_{A^c}(x)$, where $I_A(x)$ is the indicator function for a set $A$. Hence, in this limiting case, we see that $\tilde{P}_{2n}(x) = P_n(x)$, $v_\pm^{-1}(\sigma(\psi)) = \sigma(\psi) \cap \mathbb{R}^*_0$ and $\sigma(\tilde{\psi}) = \sigma(\psi)$. 


3 OLPS’S OF JACOBI, HERMITE AND LAGUERRE TYPES

For a detailed treatment of the consequences of the Transformation Theorem, see [13]. For an example of the expositions presented there, see the study of moments in [14]. Here, we will content ourselves with the application of the Transformation Theorem to several systems with MDF’s given by weight functions. If $\psi$ is an MDF which is differentiable, then $w(x) = \frac{d\psi}{dx}$ is called the weight function for $\psi$. Similarly, if $\phi$ is an SMDF which is differentiable, then $\omega(x) = \frac{d\phi}{dx}$ is called the weight function for $\phi$. If $\psi$ is an MDF, and if $\psi$ is differentiable with $d\psi/dx = w(x)$, then $\tilde{\psi}$ is differentiable with $\frac{d\tilde{\psi}}{dx} = w(v(x))$ (see [3]).

3.1 The Jacobi Class

The General Class

The monic Jacobi polynomials of parameters $\alpha > -1$ and $\beta > -1$ are denoted by $P_n^{(\alpha,\beta)}(x)$ and can be defined by the explicit formula

$$P_n^{(\alpha,\beta)}(x) = \binom{2n + \alpha + \beta}{n}^{-1} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \beta}{k} (x-1)^k (x+1)^{n-k},$$

$$n = 0, 1, 2, \ldots,$$ (3.1)

([3], equations (2.6) and (2.7), p. 144). The Jacobi MDF we denote by $\psi^{(\alpha,\beta)}_P$. It is given by

$$\frac{d\psi^{(\alpha,\beta)}_P}{dx} := \begin{cases} (1-x)^\alpha (1+x)^\beta, & \text{if } x \in (-1,1) \\ 0, & \text{otherwise} \end{cases}.$$ (3.2)

To be explicit, the orthogonality relation is

$$(P_m^{(\alpha,\beta)}, P_n^{(\alpha,\beta)})_{\psi^{(\alpha,\beta)}_P} = \int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta \, dx$$

$$= 2^{2n+\alpha+\beta+1} \binom{2n + \alpha + \beta}{n}^{-1} B(n + \alpha + 1, n + \beta + 1) \delta_{mn}$$ (3.3)

(see the discussion in [3] beginning at the bottom of page 146 and ending on the next page) where $B$ denotes the beta function, which can be given in terms of the gamma function as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and where $\delta_{mn}$ is the Kronecker delta,

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}.$$

To facilitate the reporting of the results of applying the Transformation Theorem and its consequences to the Jacobi class of systems of orthogonal polynomials, we
find it useful to make the following definitions:

\[ a := v_+^{-1}(-1) = \frac{\lambda}{2} \left( -1 + \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right) \]

and

\[ b := v_+^{-1}(1) = \frac{\lambda}{2} \left( 1 + \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right), \]

where we continue to maintain the assumption made at the beginning of Section 2 that \( \lambda \) and \( \gamma \) are fixed positive real numbers. Thus, we have assumed that \( a \) and \( b \) are the unique fixed positive real numbers having the following properties:

\[ -a = -v_+^{-1}(-1) = \frac{\lambda}{2} \left( 1 - \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right) = v_-^{-1}(1), \]

\[ -b = -v_+^{-1}(1) = \frac{\lambda}{2} \left( -1 - \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right) = v_-^{-1}(-1), \]

\[ \lambda = b - a \]

and

\[ \gamma = ab. \]

Using the Transformation Theorem and equation (3.2), we find that \( \tilde{\psi}_{\lambda}^{(\alpha,\beta)}(x) \), given by

\[
\begin{align*}
\frac{d\tilde{\psi}_{\lambda}^{(\alpha,\beta)}}{dx} &= \begin{cases} 
\frac{|x+a|^\alpha |b-x|^\alpha |x-a|^\beta |b+x|^\beta}{(b-a)^{\alpha+\beta} |x|^{\alpha+\beta}}, & \text{if } x \in (-b, -a) \cup (a, b) \\
0, & \text{if } x \in (-\infty, -b] \cup [-a, 0) \cup (0, a] \cup [b, \infty) 
\end{cases} \tag{3.4}
\end{align*}
\]

is a SMDF for each choice of parameters. The monic OLPS with respect to \( \tilde{\psi}_{\lambda}^{(\alpha,\beta)} \) we denote by \( \{\tilde{P}_m^{(\alpha,\beta)}(x)\}_{m=0}^\infty \). By Theorem 2.3.1 (C), formula (3.1) and algebra, we find the explicit formulas

\[
\tilde{P}_{2n}^{(\alpha,\beta)}(x) = \frac{1}{(2n+\alpha+\beta)_n} \frac{1}{x^n} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x+a)^{n-k}(x-b)^k(x-a)^{n-k}(x+b)^{n-k},
\]

\[ n = 0, 1, 2, \ldots, \tag{3.5} \]
\[
\tilde{P}_{2n+1}(x) = \frac{(-1/ab)^n}{(2n+\alpha+\beta)_{2n+1}} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \beta}{k} (x + a)^k (x - b)^k (x - a)^{n-k} (x + b)^{n-k},
\]
\[
n = 0, 1, 2, \ldots \quad (3.6)
\]

Considering Theorem 2.2.8 and equation (3.3), the orthogonality relation is
\[
(\tilde{P}_m^{(\alpha,\beta)}(x), \tilde{P}_n^{(\alpha,\beta)}(x))_\psi^{(\alpha,\beta)} = k_m \delta_{mn},
\]
\[
(3.7)
\]

where
\[
k_m = \begin{cases} 
(b - a)^{2j+1}2^{j+\alpha+\beta+1} \binom{2j+\alpha+\beta}{j}^{-1} B(j + \alpha + 1, j + \beta + 1), & \text{if } m = 2j \\
\left(\frac{b-a}{ab}\right)^{2j+1}2^{j+\alpha+\beta+1} \binom{2j+\alpha+\beta}{j}^{-1} B(j + \alpha + 1, j + \beta + 1), & \text{if } m = 2j + 1
\end{cases}.
\]
\[
(3.8)
\]

The Tchebycheff Polynomials of the First Kind

These polynomials can be defined by
\[
\tilde{T}_n(x) = \hat{P}_n^{(-1/2,-1/2)}(x), \quad n = 0, 1, 2, \ldots,
\]
\[
(3.9)
\]

hence the Tchebycheff polynomials of the first kind are an example from the class of Jacobi polynomials with \(\alpha = \beta = -1/2\).

\{\tilde{T}_n(x)\}_{n=0}^\infty is the monic OPS with respect to the MDF \(\psi_T\) given by
\[
\frac{d\psi_T}{dx} = \begin{cases} 
\frac{1}{\sqrt{1-x^2}}, & \text{if } x \in (-1, 1) \\
0, & \text{otherwise}
\end{cases}.
\]
\[
(3.10)
\]

To be specific, the orthogonality relation is
\[
(\tilde{T}_m, \tilde{T}_n)_{\psi_T} = \int_{-1}^{1} \tilde{T}_m(x) \tilde{T}_n(x) \frac{1}{\sqrt{1-x^2}} dx = k_n \delta_{mn},
\]
\[
(3.11)
\]

where
\[
k_n = \begin{cases} 
\pi, & \text{if } n = 0 \\
2^{1-2n} \pi, & \text{if } n \geq 1
\end{cases}.
\]
\[
(3.12)
\]

Applying the Transformation Theorem with \(\lambda = b - a\) and \(\gamma = ab\) to the monic Tchebycheff polynomials of the first kind results in a monic orthogonal Laurent
polynomial sequence \( \{ \tilde{T}_m(x) \}_{m=0}^{\infty} \) which can be defined by
\[
\tilde{T}_m(x) = \tilde{P}_m^{(-1/2,-1/2)}(x), \ m = 0, 1, 2, \ldots.
\] (3.13)

These L-polynomials, examined by Cooper and Gustafson, [5], are orthogonal with respect to the SMDF, which we denote by \( \tilde{\psi}_T \), given by
\[
\frac{d\tilde{\psi}_T}{dx} = \begin{cases} 
\frac{(b-a) |x|}{\sqrt{b^2-x^2} \sqrt{x^2-a^2}}, & \text{if } x \in (-b, -a) \cup (a, b) \\
0, & \text{if } x \in (-\infty, -b] \cup [-a, 0] \cup (0, a] \cup [b, \infty).
\end{cases}
\] (3.14)

Considering (3.11), (3.12) and Theorem 2.2.8, the orthogonality relation is
\[
\langle \tilde{T}_m, \tilde{T}_n \rangle_{\tilde{\psi}_T} = \int_{-b}^{-a} \tilde{T}_m(x) \tilde{T}_n(x) \frac{(b-a) |x|}{\sqrt{b^2-x^2} \sqrt{x^2-a^2}} dx + \int_{a}^{b} \tilde{T}_m(x) \tilde{T}_n(x) \frac{(b-a) |x|}{\sqrt{b^2-x^2} \sqrt{x^2-a^2}} dx = k_m \delta_{mn}.
\] (3.15)

where
\[
k_m = \begin{cases} 
(b-a)\pi, & \text{if } m = 0 \\
(b-a)^{2j+1} \pi, & \text{if } m = 2j \geq 2 \\
((b-a)/ab)^{2j+1} \pi, & \text{if } m = 2j + 1
\end{cases}
\] (3.16)

The Legendre Polynomials

The monic Legendre polynomials we denote by \( \hat{P}_n(x) \), \( n = 0, 1, 2, \ldots \). These polynomials can be defined by
\[
\hat{P}_n(x) = \hat{P}_n^{(0,0)}(x), \ n = 0, 1, 2, \ldots,
\] (3.17)

hence they are an instance of the Jacobi classes of polynomials. The monic Legendre polynomials thus form an OPS with respect to an MDF, which we denote by \( \psi_P \), given by
\[
\frac{d\psi_P}{dx} = \begin{cases} 
1, & \text{if } x \in (-1, 1) \\
0, & \text{otherwise}
\end{cases}
\] (3.18)

The orthogonality relation is
\[
\langle \hat{P}_m, \hat{P}_n \rangle_{\psi_P} = \int_{-1}^{1} \hat{P}_m(x) \hat{P}_n(x) dx = \frac{2^{2n+1}(n!)^4}{(2n)! (2n+1)!} \delta_{mn}.
\] (3.19)

Applying the Transformation Theorem, with \( \lambda = b - a \) and \( \gamma = ab \), to the system of monic Legendre polynomials results in a system of orthogonal Laurent polynomials. We denote the L-polynomials by \( \tilde{P}_m(x) \).
\[
\tilde{P}_m(x) = \tilde{P}_m^{(0,0)}(x), \ m = 0, 1, 2, \ldots
\] (3.20)
We denote the resulting SMDF by $\tilde{\psi}_P$, where

$$\frac{d\tilde{\psi}_P}{dx} = \begin{cases} 1, & \text{if } x \in (-b, -a) \cup (a, b) \\ 0, & \text{if } x \in (-\infty, -b) \cup [-a, 0) \cup (0, a] \cup [b, \infty) \end{cases}. \quad(3.21)$$

By Theorem 2.2.8 and equation (3.19), we see that the orthogonality relation is

$$(\tilde{P}_m, \tilde{P}_n)_{\psi_P} = \int_{-b}^{-a} \tilde{P}_m(x) \tilde{P}_n(x) \, dx + \int_{a}^{b} \tilde{P}_m(x) \tilde{P}_n(x) \, dx = k_m \delta_{mn} \quad(3.22)$$

where

$$k_m = \begin{cases} \frac{(b-a)^{2j+1}}{(2j+1)!} \frac{2^{2j+1}(j)!^2}{(2j+1)!^2}, & \text{if } m = 2j \\ \frac{(b-a)^{2j+1}}{(2j)!!} \frac{2^{2j+1}(j)!^2}{(2j+1)!!}, & \text{if } m = 2j + 1 \end{cases}. \quad(3.23)$$

### 3.2 The Generalized Hermite Class

#### The General Class

We denote the monic generalized Hermite polynomials of parameter $\alpha > -1/2$ by $\hat{H}^{(\alpha)}_n(x)$, $n = 0, 1, 2, \ldots$

$$\hat{H}^{(\alpha)}_{2k}(x) = (-1)^k k! \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k-j} (-1)^j \frac{1}{j!} x^{2j}, \quad k = 0, 1, 2, \ldots, \quad(3.24)$$

and

$$\hat{H}^{(\alpha)}_{2k+1}(x) = (-1)^k k! \sum_{j=0}^{k} \binom{k + \alpha + 1/2}{k-j} (-1)^j \frac{1}{j!} x^{2j+1}, \quad k = 0, 1, 2, \ldots, \quad(3.25)$$

(see [3], (2.43), p. 156, and (2.11), p. 145). $\{\hat{H}^{(\alpha)}_n(x)\}_{n=0}^{\infty}$ is the monic OPS with respect to the MDF $\psi_H^{(\alpha)}$ which is given by

$$\frac{d\psi_H^{(\alpha)}}{dx} = |x|^{2\alpha} e^{-x^2}, \quad x \in \mathbb{R}. \quad(3.26)$$

The orthogonality relation is

$$(\hat{H}^{(\alpha)}_m, \hat{H}^{(\alpha)}_n)_{\psi_H^{(\alpha)}} = \int_{-\infty}^{\infty} \hat{H}^{(\alpha)}_m(x) \hat{H}^{(\alpha)}_n(x) |x|^{2\alpha} e^{-x^2} \, dx = \left[ \frac{n}{2} \right] \Gamma \left( \frac{n+1}{2} \right) + \frac{1}{2} \right) \delta_{mn} \quad(3.27)$$

([3], eq. (2.45), p. 157), where $[z]$ denotes the integer part of $z$.

For each choice of $\alpha > -1/2$, the Transformation Theorem applied to the system
of monic generalized Hermite polynomials of parameter \(\alpha\) results in a system of monic orthogonal Laurent polynomials, for each choice of parameters \(\lambda > 0\) and \(\gamma > 0\). The L-polynomials, which we denote by \(\tilde{H}_m^{(\alpha)}(x)\), have the explicit expressions

\[
\tilde{H}_4k^{(\alpha)}(x) = (-1)^k k! \lambda^{2k} \frac{1}{x^{2k}} \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k - j} \frac{(\gamma^2)^j}{j! \lambda^{2j}} (x^2 - \gamma)^{2j} x^{2(k-j)},
\]

\(k = 0, 1, 2, \ldots, \) \(3.28\)

\[
\tilde{H}_{4k+1}^{(\alpha)}(x) = (-1)^k k! \left(\frac{\lambda}{\gamma}\right)^{2k+1} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k - j} \frac{(\gamma^2)^j}{j! \lambda^{2j+1}} (x^2 - \gamma)^{2j+1} x^{2(k-j)},
\]

\(k = 0, 1, 2, \ldots, \) \(3.29\)

\[
\tilde{H}_{4k+2}^{(\alpha)}(x) = (-1)^k k! \lambda^{2k+1} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \binom{k + \alpha + 1/2}{k - j} \frac{(\gamma^2)^j}{j! \lambda^{2j+1}} (x^2 - \gamma)^{2j+1} x^{2(k-j)},
\]

\(k = 0, 1, 2, \ldots, \) \(3.30\)

and

\[
\tilde{H}_{4k+3}^{(\alpha)}(x) = (-1)^k k! \left(\frac{\lambda}{\gamma}\right)^{2k+1} \frac{1}{x^{2(k+1)}} \sum_{j=0}^{k} \binom{k + \alpha + 1/2}{k - j} \frac{(\gamma^2)^j}{j! \lambda^{2j+1}} (x^2 - \gamma)^{2j+1} x^{2(k-j)},
\]

\(k = 0, 1, 2, \ldots, \) \(3.31\)

given by equations (3.24) and (3.25) and Theorem 2.3.1 (C). According to the Transformation Theorem and equation (3.26), \(\{\tilde{H}_m^{(\alpha)}(x)\}_{m=0}^{\infty}\) is the monic OLPS with respect to a SMDF, which we denote by \(\tilde{\psi}_H^{(\alpha)}\), defined by

\[
\frac{dx \tilde{\psi}_H^{(\alpha)}}{d_\gamma} = \frac{1}{\lambda^{2a}} \left| x^2 - \gamma + \frac{\gamma^2}{\lambda^2}\right|^\alpha \exp \left(-\frac{1}{\lambda^2} \left(x^2 - 2\gamma + \frac{\gamma^2}{\lambda^2}\right)\right), \ x \in \mathbb{R}^+ \cup \mathbb{R}^- . \ (3.32)
\]

By (3.26), (3.27) and Theorem 2.2.8, the orthogonality relation is

\[
(\tilde{H}_m^{(\alpha)}, \tilde{H}_n^{(\alpha)})_{\tilde{\psi}_H^{(\alpha)}} = k_m \delta_{mn} , \quad (3.33)
\]

where

\[
k_m = \begin{cases} 
\lambda^{2j+1} \left(\frac{j+1}{2}\right)! \Gamma \left(\frac{j+1}{2} + \alpha + \frac{1}{2}\right), & \text{if } m = 2j \\
\frac{\lambda}{\gamma} \left(\frac{j}{2}\right)^{2j+1} \left(\frac{j+1}{2}\right)! \Gamma \left(\frac{j+1}{2} + \alpha + \frac{1}{2}\right), & \text{if } m = 2j + 1 .
\end{cases} \ (3.34)
\]

**The Hermite Polynomials**

The monic *Hermite polynomials*, which we denote by \(\hat{H}_n(x)\), are the monic generalized Hermite polynomials of parameter \(\alpha = 0\):

\[
\hat{H}_n(x) = \tilde{H}_n^{(0)}(x), \ n = 0, 1, 2, \ldots . \ (3.35)
\]
\( \{ \tilde{H}_n(x) \}_{n=0}^{\infty} \) is the monic OPS with respect to the MDF \( \psi_H \) which is given by
\[
\frac{d\psi_H}{dx} = e^{-x^2}, \ x \in \mathbb{R}.
\] (3.36)

The orthogonality relation is
\[
(\tilde{H}_m, \tilde{H}_n)_{\psi_H} = \int_{-\infty}^{\infty} \tilde{H}_m(x) \tilde{H}_n(x) e^{-x^2} \, dx = \frac{n!}{2n} \sqrt{\pi} \delta_{mn},
\] (3.37)

We apply the Transformation Theorem to obtain a system of orthogonal Laurent polynomials. We denote the L-polynomials by \( \tilde{H}_m(x) \), and, considering (3.35),
\[
\tilde{H}_m(x) = \tilde{H}_{m(0)}(x) = \tilde{H}_{m(0)}(0), \quad m = 0, 1, 2, \ldots.
\] (3.38)

\( \{ \tilde{H}_m(x) \}_{m=0}^{\infty} \) is the monic OLPS with respect to the SMDF \( \tilde{\psi}_H = \tilde{\psi}_{H(0)} \), given by
\[
\frac{d\tilde{\psi}_H}{dx} = e^{-\frac{1}{\lambda}(x^2 - 2\gamma + \frac{\gamma^2}{4})}, \ x \in \mathbb{R}^- \cup \mathbb{R}^+.
\] (3.39)

according to (3.32) with \( \alpha = 0 \). The orthogonality relation is
\[
(\tilde{H}_m, \tilde{H}_n)_{\tilde{\psi}_H} = \int_{-\infty}^{0} \tilde{H}_m(x) \tilde{H}_n(x) e^{-\frac{1}{\lambda}\left(x^2 - 2\gamma + \frac{\gamma^2}{4}\right)} \, dx + \int_{0}^{\infty} \tilde{H}_m(x) \tilde{H}_n(x) e^{-\frac{1}{\lambda}\left(x^2 - 2\gamma + \frac{\gamma^2}{4}\right)} \, dx = k_m \delta_{mn},
\] (3.40)

where
\[
k_m = \begin{cases} 
\lambda^{2j+1} j! \sqrt{\pi}, & \text{if } m = 2j \\
(\frac{1}{\gamma})^{2j+1} \frac{1}{2j+1} \sqrt{\pi}, & \text{if } m = 2j + 1.
\end{cases}
\] (3.41)

### 3.3 The Generalized Laguerre Class

The systems of orthogonal polynomials associated with the names of Jacobi, Hermite, and Laguerre are collectively called the classical orthogonal polynomials. In this section we finish our treatment of applying the Transformation Theorem and some of its consequences to the systems of classical polynomials by considering those polynomials associated with Laguerre.

**A Direct Application of the Transformation Theorem**

The monic Sonine-Laguerre or generalized Laguerre polynomials of parameter \( \alpha > -1 \) are denoted by \( \tilde{L}_n^{(\alpha)}(x) \). These polynomials can be defined by the explicit
expressions

\[ \hat{L}_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \frac{(n + \alpha)}{n - k} \frac{(-x)^k}{k!}, \quad n = 0, 1, 2, \ldots \] (3.42)

\[ ([3], (2.11) and (2.12), p. 154), \text{ and they are orthogonal with respect to the MDF } \psi_L^{(\alpha)}, \text{ given by} \]

\[ \frac{d\psi_L^{(\alpha)}}{dx} = \begin{cases} x^\alpha e^{-x}, & \text{if } x \in \mathbb{R}^+ \\ 0, & \text{otherwise} \end{cases}. \] (3.43)

The orthogonality relation is

\[ (\hat{L}_m^{(\alpha)}, \hat{L}_n^{(\alpha)})_{\psi_L^{(\alpha)}} = \int_0^\infty \hat{L}_m^{(\alpha)}(x) \hat{L}_n^{(\alpha)}(x) x^\alpha e^{-x} \, dx = n! \Gamma(n + \alpha + 1) \delta_{mn} \] (3.44)

\[ ([3], (2.18), p. 148, and (2.12), p. 145). \]

As mentioned at the end of Section 2, applying the Transformation Theorem to an MDF results in an SMDF, for each choice of \( \lambda > 0 \) and \( \gamma > 0 \), with part of the spectrum in the negative reals and part in the positive reals. This raises serious doubts as to the appropriateness of the resulting SMDF as an analogue to the corresponding MDF whose spectrum is contained, say, in the non-negative reals. One may consider such MDF’s as examples of the limitations of this transformation.

The generalized Laguerre class here provides a collection of such MDF’s, since we see that the resulting SMDF, which we denote by \( \psi_L^{(\alpha)} \), is given by

\[ \frac{d\psi_L^{(\alpha)}}{dx} = \begin{cases} \left( \frac{1}{\lambda^2} (x - \frac{\gamma^2}{x}) \right)^\alpha e^{-\frac{1}{\lambda^2}(x-\frac{\gamma^2}{x})}, & \text{if } x \in (-\sqrt{\gamma}, 0) \cup (\sqrt{\gamma}, \infty) \\ 0, & \text{if } x \in (-\infty, -\sqrt{\gamma}] \cup [0, \sqrt{\gamma}] \end{cases}. \] (3.45)

According to (3.43) and the Transformation Theorem. It is evident from the definitions that the spectrum \( \sigma(\psi_L^{(\alpha)}) \) is \([ -\sqrt{\gamma}, 0 ) \cup (\sqrt{\gamma}, \infty) \), while \( \sigma(\psi_L^{(\alpha)}) = [0, \infty) \). We will return to the problem of Laguerre-like strong moment distributions at the end of this section.

Regardless of our views about the question of appropriateness in this case, we can proceed to transfer the formulas for the monic generalized Laguerre polynomials of parameter \( \alpha > -1 \) over to the corresponding OLPS given by the Transformation Theorem. We denote the L-polynomials by \( \tilde{L}_m^{(\alpha)}(x) \), and we find the explicit expressions

\[ \tilde{L}^{(\alpha)}_{2n}(x) = (-1)^n n! \lambda^n \frac{1}{x^n} \sum_{k=0}^{n} \frac{(n + \alpha)}{n - k} \frac{(-1)^k}{k!\lambda^k} (x^2 - \gamma)^k x^{n-k}, \quad n = 0, 1, 2, \ldots \] (3.46)

and

\[ \tilde{L}^{(\alpha)}_{2n+1}(x) = n! \left( \frac{\lambda}{\gamma} \right)^n \frac{1}{x^{n+1}} \sum_{k=0}^{n} \frac{(n + \alpha)}{n - k} \frac{(-1)^k}{k!\lambda^k} (x^2 - \gamma)^k x^{n-k}, \quad n = 0, 1, 2, \ldots \] (3.47)
by (3.42) and Theorem 2.3.1 (C). Considering (3.45), the inner products (3.44) and Theorem 2.2.8, the orthogonality relation is

\[
(\tilde{L}_m^{(\alpha)} , \tilde{L}_n^{(\alpha)})_{\varphi^{(\alpha)}} = \int_{-\sqrt{\gamma}}^{0} \tilde{L}_m^{(\alpha)}(x)\tilde{L}_n^{(\alpha)}(x) \left( \frac{1}{\lambda} (x - \frac{\gamma}{x}) \right)^\alpha e^{-\frac{x}{2}(x-\tilde{\gamma})} dx \\
+\int_{\sqrt{\gamma}}^{\infty} \tilde{L}_m^{(\alpha)}(x)\tilde{L}_n^{(\alpha)}(x) \left( \frac{1}{\lambda} (x - \frac{\gamma}{x}) \right)^\alpha e^{-\frac{x}{2}(x-\tilde{\gamma})} dx
\]

\[= k_m \delta_{mn}, \]

where

\[k_m = \begin{cases} \lambda^{2j+1} (\alpha + 1)_j \Gamma(\alpha + 1), & \text{if } m = 2j \\ \left( \frac{\lambda}{\gamma} \right)^{2j+1} (\alpha + 1)_j \Gamma(\alpha + 1), & \text{if } m = 2j + 1. \end{cases} \tag{3.49} \]

**A Laguerre Class Related to the Generalized Hermite Class**

Let \(\alpha > -1/2\) and define \(\{A_m^{(\alpha)}(x)\}_{m=0}^{\infty}\) by

\[A_{2n}^{(\alpha)}(x^2) := \tilde{H}_{4n}^{(\alpha)}(x), \ n = 0, 1, 2, \ldots, \tag{3.50} \]

and

\[A_{2n+1}^{(\alpha)}(x^2) := \tilde{H}_{4n+3}^{(\alpha)}(x), \ n = 0, 1, 2, \ldots. \tag{3.51} \]

By (3.28),

\[A_{2n}^{(\alpha)}(x) = (-1)^n n! \lambda^{2n} \frac{1}{x^n} \sum_{j=0}^{n} \binom{n + \alpha - 1/2}{n - j} \frac{(-1)^j}{j!} \lambda^{2j} (x - \gamma)^{2j} x^{n-j}, \ 
\]

\[n = 0, 1, 2, \ldots, \tag{3.52} \]

and by (3.31),

\[A_{2n+1}^{(\alpha)}(x) = (-1)^{n+1} n! \left( \frac{\lambda}{\gamma} \right)^{2n+1} \frac{1}{x^{n+1}} \sum_{j=0}^{n} \binom{n + \alpha + 1/2}{n - j} \frac{(-1)^j}{j!} \lambda^{2j+1} (x - \gamma)^{2j+1} x^{n-j}, \]

\[n = 0, 1, 2, \ldots. \tag{3.53} \]

Inspection of (3.52) and (3.53) shows that \(A_m^{(\alpha)}(x)\) is a monic L-polynomial of L-degree \(m\), for each \(m = 0, 1, 2, \ldots. \)

Next, define \(\phi_A^{(\alpha)}(x)\) by

\[
\frac{d\phi_A^{(\alpha)}}{dx} := \begin{cases} \frac{1}{\lambda} x^{-\frac{\gamma}{\lambda}} |x - 2\gamma + \frac{\gamma^2}{x}|^\alpha \exp \left( -\frac{1}{\lambda} x - 2\gamma + \frac{\gamma^2}{x} \right), & \text{if } x \in \mathbb{R}^+ \\ 0, & \text{if } x \in \mathbb{R}^- \end{cases} \tag{3.54} \]

Considering (3.54), \(\phi_A^{(\alpha)}\) is a SMDF with spectrum \(\sigma(\phi_A^{(\alpha)}) = \mathbb{R}^+\), whose moments
are
\[ \mu_l(\phi_A^{(\alpha)}) = \mu_{2l}(\psi_H^{(\alpha)}), \quad l = 0, \pm 1, \pm 2, \ldots, \]  
(3.55)

by the change of variables \( x^2 \to x \) in
\[ \mu_{2l}(\psi_H^{(\alpha)}) = 2 \int_0^\infty x^{2l} \frac{1}{\lambda^{2\alpha}} \left| x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right|^\alpha e^{-\frac{1}{\lambda^2} \left( x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right)} \, dx. \]

If \( k, l, m \) and \( n \) are non-negative integers such that \( A_m^{(\alpha)}(x^2) = \overline{H_k}^{(\alpha)}(x) \) and \( A_n^{(\alpha)}(x^2) = \overline{H_l}^{(\alpha)}(x) \), then, by linearity of the integral and (3.55),
\[ (A_m^{(\alpha)}, A_n^{(\alpha)})_{\phi_A^{(\alpha)}} = (\overline{H_k}^{(\alpha)}, \overline{H_l}^{(\alpha)})_{\psi_H^{(\alpha)}}. \]  
(3.56)

Hence, \( \{A_m^{(\alpha)}(x)\}_{m=0}^\infty \) is the monic OLPS with respect to \( \phi_A^{(\alpha)} \), the orthogonality relation being
\[ (A_m^{(\alpha)}, A_n^{(\alpha)})_{\psi_A^{(\alpha)}} = k_m \delta_{mn}, \]  
(3.57)

where
\[ k_m = \begin{cases} \lambda^{4j+1}(j!) \Gamma \left( \left[ \frac{2j+1}{2} \right] + \alpha + \frac{1}{2} \right), & \text{if } m = 2j \\ (\frac{\lambda}{\gamma})^{4j+3} \left( \left[ \frac{2j+1}{2} \right] ! \right) \Gamma \left( j + \alpha + \frac{3}{2} \right), & \text{if } m = 2j + 1. \end{cases} \]  
(3.58)

REFERENCES