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LAGUERRE-TYPE MOMENTS VIA LAPLACE TRANSFORM

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ABSTRACT. Connections are revealed between the n-th moments $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s)$ of parameter $s > 0$ associated with the modified Laguerre distribution having weight function $w(t) := F(t)e^{-st}$ on the non-negative reals and the Laplace transform $f(s)$ of the modifying function $F(t)$. In particular, $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = (-1)^n \frac{d^n}{ds^n} f(s)$ if n is a non-negative integer. For negative n, there is a corresponding iterated integral formula. Specific examples of moment and strong moment distributions are given, and transforms related to other classical systems are introduced.

Starting points for basic definitions and examples of moment sequences include Chihara's text in orthogonal polynomials [2] and the survey [7] of strong moment theory by Jones and Njåstad, which also provides an extensive bibliography for further reference. In addition, a treatise on the elements of real analysis and the NBS handbook [1] may be useful to the reader.

The Laplace transform $f(s)$ of a function $F(t)$,

$$f(s) = \mathcal{L}\left\{F(t)\right\}(s) := \int_0^\infty F(t)e^{-st}\,dt, \tag{0.1}$$

is reminiscent of the moments $\mu_n^{\mathcal{L}}$ of a Laguerre distribution,

$$\mu_n^{\mathcal{L}} := \int_0^\infty t^n e^{-t}\,dt,\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.2}$$

By introducing a parameter $s > 0$, modifying the weight function slightly to e^{-st}, the corresponding moments $\mu_n^{\mathcal{L}}(s)$ are Laplace transforms.

$$\mu_n^{\mathcal{L}}(s) := \int_0^\infty t^n e^{-st}\,dt,\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.3}$$

The integrand $t^n e^{-st}$ and its partial derivatives are uniformly continuous, hence

$$\frac{d}{ds}\int_0^\infty t^n e^{-st}\,dt = \int_0^\infty \frac{\partial}{\partial s}\left(t^n e^{-st}\right)\,dt = \int_0^\infty \left(-t^{n+1}e^{-st}\right)\,dt;$$

that is, $\frac{d}{ds}\mu_n^{\mathcal{L}}(s) = -\mu_{n+1}^{\mathcal{L}}(s)$. By Iteration,

$$\mu_n^{\mathcal{L}}(s) = (-1)^n \frac{d^n}{ds^n}\mu_0^{\mathcal{L}}(s);$$

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that is,

$$\mu_n^{\mathcal{L}}(s) = (-1)^n \frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}},\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.4}$$

Taking $s = 1$, we recover the formula for the n-th moment of the Laguerre distribution, $\mu_n^{\mathcal{L}} = n!$.

1. Laguerre-Type Moments Via Laplace Transform

To apply the previous ideas to strong moment bisequences such as

$$\int_0^\infty t^n t^{-1} e^{-t-1/t} dt,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots, \tag{1.1}$$

insert a real-valued modifying function $F(t)$ with domain containing the non-negative real numbers and consider the integrals

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty t^n F(t) e^{-st} dt. \tag{1.2}$$

Under some restrictions on the integrands, elementary methods of calculus imply

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_{n-1}^{\mathcal{L}}\left\{F(t)\right\}(s), \tag{1.3a}$$

and

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = \int_s^\infty \mu_{n+1}^{\mathcal{L}}\left\{F(t)\right\}(x)\ dx, \tag{1.3b}$$

and, recalling the definitions (0.1) and (1.2),

$$\mu_0^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty F(t) e^{-st}\ dt = f(s),$$

Thus, considering Equations (1.3), we have

Theorem 1.1. *Let $s > 0$. Suppose $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty t^n F(t) e^{-st} dt$ is the n-th moment of a moment distribution function ψ and $f(s)$ is the Laplace transform of $F(t)$. Then*

$$\mu_0^{\mathcal{L}}\left\{F(t)\right\}(s) = f(s). \tag{1.4a}$$

If $t^k F(t) e^{-st}$ is uniformly continuous for $0 \le k \le n$, then

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = (-1)^n \frac{d^n}{ds^n} f(s). \tag{1.4b}$$

If, additionally, ψ is a strong moment distribution function and $t^k F(t) e^{-st}$ is uniformly continuous for $-n \le k \le 0$, then

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = I^{|n|} f(s), \tag{1.4c}$$

where $I^1 f(s) := \int_s^\infty f(x)\ dx$ and $I^{|n|} f(s) := \int_s^\infty I^{|n|-1} f(x)\ dx$ for $|n| > 1$.

Remark 1.2. As can often be the case for such theorems, the uniform continuity conditions are stronger than those which are strictly necessary.

2. Example

Reconsider (1.1), which can be rewritten as

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\int_0^\infty t^{n-1}e^{-t-1/t}dt.$$

These are the moments of a strong moment distribution function, and the Laplace transform of $t^{-1}e^{-1/t}$ is known to be $2K_0(2\sqrt{s})$, where $K_\nu(z)$ is a modified Bessel function of the second kind. By Theorem (1.1),

$$\mu_0^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=2K_0(2\sqrt{s}),\tag{2.1a}$$

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=(-1)^n\frac{d^n}{ds^n}2K_0(2\sqrt{s})\ \mathit{for}\ n=1,\ 2,\ 3,\ \ldots,\tag{2.1b}$$

and

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=I^{|n|}2K_0(2\sqrt{s})\ \mathit{for}\ n=-1,\ -2,\ -3,\ \ldots,\tag{2.1c}$$

where $I^1 2K_0(2\sqrt{s}):=\int_s^\infty 2K_0(2\sqrt{x})\,dx$ and $I^{|n|}2K_0(2\sqrt{s}):=\int_s^\infty I^{|n|-1}2K_0(2\sqrt{x})\,dx$ for $|n|>1$.

Using (2.1a), (1.3a), elementary calculus, and the known fact that $\frac{d}{dz}\left(\frac{1}{z^n}K_n(z)\right)=\frac{-1}{z^n}K_{n+1}(z)$, an induction argument shows that $\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=\frac{2}{\sqrt{s^n}}K_n(2\sqrt{s})$ for $n=0,\ 1,\ 2,\ \ldots$. Also, by a change of variables $t\to 1/(su)$, it follows that $\mu_{-n}^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=s^n\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)$. Subsequently,

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=\int_0^\infty t^{n-1}e^{-st-1/t}dt=\frac{2}{\sqrt{s^n}}K_{|n|}(2\sqrt{s})\quad \mathit{for}\ n=0,\ \pm1,\ \pm2,\ \ldots.\tag{2.2}$$

A greater symmetry in these moments results when $s=1$; in particular,

$$\mu_{-n}^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1),\tag{2.3}$$

since, in this case,

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\int_0^\infty t^{n-1}e^{-t-1/t}dt=2K_{|n|}(2)\quad \mathit{for}\ n=0,\ \pm1,\ \pm2,\ \ldots.\tag{2.4}$$

Table 1 gives numerical values for these moments.

3. Further Examples and Transforms

A few formulas, including (2.2), for the moments of modified Laguerre distributions which can be obtained via Theorem (1.1) are given in Table 2.

Other integral transforms can be employed to obtain formulas for moment sequences, but more than cursory descriptions is beyond the goals of this article. Several extensions of the one just explicated are possible. For distributions associated with a weight function $w(t)$ on an interval I, investigations lead back to transforms similar to Laplace; for example,

$$\mathcal{M}\left\{F(t)\right\}(s):=\int_I F(t)w(t)e^{(1-s)t}\ dt,\tag{3.1a}$$

TABLE 1. Numerical Values of the Moments

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1) = \int_0^\infty t^{n-1}e^{-t-1/t}dt = 2K_{|n|}(2).$$

n	$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)$
0	0.22779
±1	0.27973
±2	0.50752
±3	1.29477
±4	4.39183
±5	18.86210

Remark: The recursion $\mu_{n+1} = n\mu_n + \mu_{n-1}$ holds for $n > 1$.

TABLE 2. Moments $\mu_n = \int_0^\infty t^n F(t)e^{-st}dt$, Parameter $s > 0$.

$F(t)$	μ_n	*Comments*
$t^{-1}e^{-1/t}$	$\frac{2}{\sqrt{s^n}}K_{\lvert n\rvert}(2\sqrt{s})$	modified Bessel function $K_\nu(z)$
$\frac{1}{\sqrt{\pi t}}e^{-1/t}$	$(-1)^n\frac{d^n}{ds^n}\frac{e^{-2\sqrt{s}}}{\sqrt{s}}$	$n \geq 0$
	$I^{\lvert n\rvert}\frac{e^{-2\sqrt{s}}}{\sqrt{s}}$	$n < 0$, see Note below
$\frac{k}{2\sqrt{\pi t^3}}e^{\frac{-k^2}{4t}}$	$(-1)^n\frac{d^n}{ds^n}e^{-k\sqrt{s}}$	$0 < k < \infty,\ n \geq 0$
	$I^{\lvert n\rvert}e^{-k\sqrt{s}}$	$n < 0$, see Note below
$[1 + \lvert t-k\rvert/(t-k)]/2$	$\frac{e^{-sk}}{s}\sum_{j=0}^{n}\binom{n}{j}k^j s^{j-n}$	$0 \leq k < \infty,\ n \geq 0$
	$I^{\lvert n\rvert}\frac{e^{-sk}}{s}$	$0 < k < \infty,\ n < 0$, see Note below
1	$\frac{n!}{s^{n+1}}$	$n \geq 0$
t^m	$\frac{\Gamma(m+n+1)}{s^{m+n+1}}$	$m+n+1 > 0$, gamma function $\Gamma(z)$

Note: $I^1 f(s) := \int_s^\infty f(x)\ dx$ and $I^{|n|}f(s) := \int_s^\infty I^{|n|-1}f(x)\ dx$ for $|n| > 1$.

and moments

$$\mu_n^{\mathcal{M}}\{F(t)\}(s) := \int_I t^n F(t)w(t)e^{(1-s)t}\ dt, \tag{3.1b}$$

which, under some restrictions, satisfy

$$\mu_n^{\mathcal{M}}\{F(t)\}(s) = -\frac{d}{ds}\mu_{n-1}^{\mathcal{M}}\{F(t)\}(s) \tag{3.1c}$$

and a corresponding integral formula. When $w(t) = e^{-t}$ on $I = [0,\infty)$, $\mathcal{M} = \mathcal{L}$ and $\mu_n^{\mathcal{M}}\{F(t)\}(s) = \mu_n^{\mathcal{L}}\{F(t)\}(s)$, as desired.

With $w(t) = (1-t)^\alpha(1+t)^\beta$ on $I = (-1, 1)$, the transform and moments (3.1) in the classical Jacobi case are

$$\mathcal{P}^{(\alpha,\beta)}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)(1-t)^\alpha(1+t)^\beta e^{(1-s)t}dt \quad (\textit{Jacobi}) \tag{3.2a}$$

and

$$\mu_n^{\mathcal{P}^{(\alpha,\beta)}}\left\{F(t)\right\}(s) := \int_{-1}^{1} t^n F(t)(1-t)^\alpha(1+t)^\beta e^{(1-s)t}dt. \tag{3.2b}$$

For example, various choices of the parameters α and β give transforms associated with the Legendre and Chebyshev distributions,

$$\mathcal{P}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)e^{(1-s)t}dt \quad (\textit{Legendre}) \tag{3.3}$$

and

$$\mathcal{T}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)\frac{e^{(1-s)t}dt}{\sqrt{1-t^2}} \quad (\textit{Chebyshev}). \tag{3.4}$$

Although useful differential and integral formulas for the moments result, symmetry of the distributions in these two cases is lost.

As hinted in the previous paragraph, it may be desirable to consider other transformations. One may wish to preserve certain aspects of the distributions and moments that may not carry over using (3.1). For example,

$$\mathcal{H}\left\{F(t)\right\}(s) := \int_{-\infty}^{\infty} F(t)e^{-st^2}dt \quad (\textit{Hermite}) \tag{3.5}$$

can be examined with one eye on the lookout for the relationship to the Laguerre case involving the Laplace transform. Here, under certain conditions on the modifying function $F(t)$, the relation

$$\mu_n^{\mathcal{H}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_{n-2}^{\mathcal{H}}\left\{F(t)\right\}(s), \tag{3.6}$$

between the moments

$$\mu_n^{\mathcal{H}}\left\{F(t)\right\}(s) := \int_{-\infty}^{\infty} t^n F(t)e^{-st^2}dt \tag{3.7}$$

exists, and the symmetry of the Hermite distribution is maintained when $F(t)$ is an even function.

4. Concluding Remarks

For the theory of orthogonal functions, two apparent contributions of the methods described above are (1) differential and integral formulas for moments and (2) versatile, systematic and elementary ways of modifying moment functionals, in particular, for the construction of positive-definite strong moment functionals. Although the approach (1) is fundamentally different from those investigated in the past by Hagler, the present article continues the author's work on the theme (2), as evident in [3, 4, 5, 6].

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