4.1. The Poisson Class

The Charlier distribution $\psi_C^{(\alpha)}$ of parameter $\alpha \neq 0$ is defined by

$$
\psi_C^{(\alpha)}(x) := e^{-\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} I_{[n,\infty)}(x), \ x \in \mathbb{R}.
$$

(4.1.1)

It is easy to verify that $\psi_C^{(\alpha)}$ is a discrete MDF with jumps $j_n = \frac{\alpha^n}{n!} e^{-\alpha}$ at $x_n = n$ (for $n = 0, 1, 2, \ldots$) if and only if $\alpha > 0$, where the moments are

$$
\mu_n(\psi_C^{(\alpha)}) = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} k^n, \ n = 0, 1, 2, \ldots
$$

(4.1.2)

In the case $\alpha > 0$, $\psi_C^{(\alpha)}$ is the Poisson distribution function of probability theory and is called the Poisson MDF of parameter $\alpha > 0$.

Much is known about the Poisson MDF. We denote by $\{\tilde{C}_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ the monic OPS. An explicit representation for these polynomials is

$$
\tilde{C}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k!(-\alpha)^{n-k}, \ n = 0, 1, 2, \ldots
$$

(4.1.3)
([Chi], eq. (1.2), p.170), where \( \binom{x}{0} := 1 \) and
\[
\binom{x}{k} := \frac{x(x-1)(x-2) \cdots (x-k+1)}{k!}, \quad k = 1, 2, 3, \ldots.
\]

The orthogonality relation for these polynomials is
\[
\left( \hat{C}_m^{(\alpha)}, \hat{C}_n^{(\alpha)} \right)_{\psi_{\alpha}^{(\alpha)}} = \alpha^n n! \delta_{mn} \tag{4.1.4}
\]
([Chi], eq. (1.3), p. 170), where \( \delta_{mn} \) is the “Kronecker delta”
\[
\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}.
\]

The Poisson monic OPS is also defined by the generating function
\[
e^{-\alpha r(1+r)^x} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{C}_n^{(\alpha)}(x) r^n \tag{4.1.5}
\]
([Chi], eq. (1.1), p. 170), by the fundamental recurrence formula
\[
\hat{C}_n^{(\alpha)}(x) = (x-n+1-\alpha)\hat{C}_{n-1}^{(\alpha)}(x) - \alpha(n-1)\hat{C}_{n-2}^{(\alpha)}(x), \quad n = 1, 2, 3, \ldots \tag{4.1.6}
\]
([Chi], eq. (1.4), p. 170), and by the finite difference analogue of Rodrigues’
formula
\[
\hat{C}_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(x+1)}{\alpha^x} (\Delta^n Y_n)(x), \quad n = 0, 1, 2, \ldots, \tag{4.1.7}
\]
with
\[
Y_n(x) = \frac{\alpha^x}{\Gamma(x-n+1)}
\]
([Chi], eq. (3.2), p. 160), where $\Gamma$ is the function commonly known as the “Gamma function”.

We now reap the harvest of our labor of the previous chapters for this special case of the Poisson MDF. By equation (4.1.1) and Theorem 3.1.1,

$$
\tilde{\psi}^{(\alpha)}_C(x) = e^{-\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n}{n! v'(v^{-1}(n))} I_{v^{-1}(n), \infty}(x), \quad x \in \mathbb{R}^\pm,
$$

(4.1.8)
is a discrete SMDF for each choice of $\alpha > 0$. An expression for the moments of $\tilde{\psi}^{(\alpha)}_C$ can be found by combining equation (4.1.2) with Theorem 3.3.5; for example, we can find

$$
\mu_0(\tilde{\psi}^{(\alpha)}_C) = \lambda
$$

(4.1.9)
in this way. However, we can also find

$$
\mu_l(\tilde{\psi}^{(\alpha)}_C)
= e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k! v'(v^{-1}(k))} \left(v^{-1}(k)\right)^l + e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k! v'(v^{-1}_+(k))} \left(v^{-1}_+(k)\right)^l,
$$

(4.1.10)
for $l = 0, \pm 1, \pm 2, \ldots$, directly by considering equation (4.1.8).

The L-polynomials of the monic OLPS $\{\tilde{C}_m^{(\alpha)}(x)\}_{m=0}^{\infty}$ with respect to $\tilde{\psi}^{(\alpha)}_C$ have the explicit representations given by Theorem 2.3.1 (C) and equation (4.1.3)

$$
\tilde{C}^{(\alpha)}_{2n}(x) = \lambda^n \sum_{k=0}^{n} \binom{n}{k} \left(v(x)\right)^k (-\alpha)^{n-k}, \quad n = 0, 1, 2, \ldots,
$$

(4.1.11)
\[ C_{2n+1}^{(\alpha)}(x) = \left( -\frac{\lambda}{\gamma} \right)^n \frac{1}{x} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{v(x)}{k} \right) k!(-\alpha)^{n-k}, \quad n = 0, 1, 2, \ldots \quad (4.1.12) \]

The orthogonality relation, given by Theorem 2.2.8 and equation (4.1.4), is

\[ \left( \widehat{C}_m^{(\alpha)}, \widehat{C}_n^{(\alpha)} \right)_{\psi_C^{(\alpha)}} = k_m \delta_{mn} \quad (4.1.13) \]

where

\[ k_m = \begin{cases} \lambda^{2j+1} \alpha^j \ j!, & \text{if } m = 2j \\ \left( \frac{\lambda}{\gamma} \right)^{2j+1} \alpha^j \ j!, & \text{if } m = 2j + 1 \end{cases} \quad (4.1.14) \]

By Theorem 3.9.1 and equation (4.1.5), as a formal power series,

\[ e^{-\alpha \lambda r^2} \left( 1 + \lambda r^2 \right)^{v(x)} + \frac{r}{x} e^{\alpha \frac{r^2}{\gamma}} \left( 1 - \frac{\lambda}{\gamma} r^2 \right)^{\frac{v(x)}{\gamma}} = \sum_{m=0}^{\infty} b_m \widehat{C}_m^{(\alpha)}(x) r^m \quad (4.1.15) \]

where

\[ b_{2n+1} = b_{2n} = \frac{1}{n!} \quad (4.1.16) \]

By Theorem 3.7.2 and the difference equation (4.1.7),

\[ \widehat{C}_{2n}^{(\alpha)}(x) = \left( -\frac{\lambda}{\gamma} \right)^n \Gamma \left( v(x) + 1 \right) \frac{\Delta^n Y_n(v(x))}{\alpha^{v(x)}} \quad (4.1.17) \]

and

\[ \widehat{C}_{2n+1}^{(\alpha)}(x) = \frac{\lambda^n \Gamma(v(x) + 1)}{\gamma^n \alpha^{v(x)} x} \Delta^n Y_n(v(x)) \quad (4.1.18) \]
where

\[ Y_n(x) = \frac{\alpha^x}{\Gamma(x - n + 1)}. \]

By Theorem 3.5.3 and the recurrence formula (4.1.6), \( \{\tilde{C}_m^{(\alpha)}(x)\}_{m=0}^{\infty} \) satisfies the fundamental recurrence formula

\[ \tilde{C}_m^{(\alpha)}(x) = \alpha_m \tilde{C}_{m-1}^{(\alpha)}(x) + (x(-1)^m + \beta_m)\tilde{C}_{m-2}^{(\alpha)}(x) + \gamma_m \tilde{C}_{m-3}^{(\alpha)}(x) + \delta_m \tilde{C}_{m-4}^{(\alpha)}(x), \quad m = 2, 3, 4, \ldots, \quad (4.1.19) \]

where we set \( \tilde{C}_k^{(\alpha)}(x) \equiv 0 \) for \( k < 0 \) and where the coefficients \( \alpha_m, \beta_m, \gamma_m \) and \( \delta_m \) are given by

\[
\begin{align*}
\alpha_{2n} &= (-\gamma)^n, & \alpha_{2n+1} &= 0, \quad (4.1.20) \\
\beta_{2n} &= -\lambda(n - 1 + \alpha), & \beta_{2n+1} &= \frac{\lambda}{\gamma}(n - 1 + \alpha), \quad (4.1.21) \\
\gamma_{2n} &= 0, & \gamma_{2n+1} &= (-\frac{1}{\gamma})^n, \quad (4.1.22) \\
\delta_{2n} &= -\lambda^2(n - 1)\alpha \text{ and} & \delta_{2n+1} &= -\left(\frac{\lambda}{\gamma}\right)^2(n - 1)\alpha, \quad (4.1.23)
\end{align*}
\]

for \( n = 1, 2, 3, \ldots \).

4.2. The Jacobi Class

4.2.1. The General Class. The monic Jacobi polynomials of parameters \( \alpha > -1 \) and \( \beta > -1 \) are denoted by \( \tilde{P}_n^{(\alpha, \beta)}(x) \) and can be defined by the Rodrigues’ type formula
\[ P_n^{(\alpha,\beta)}(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n} \left( \rho^n(x) w(x) \right), \quad n = 0, 1, 2, \ldots, \quad (4.2.1.1) \]

where
\[ K_n = (-1)^n (n!)^2 \left( \frac{2n + \alpha + \beta}{n} \right), \quad (4.2.1.2) \]
\[ \rho(x) = 1 - x^2 \quad (4.2.1.3) \]

and
\[ w(x) = (1 - x)^\alpha (1 + x)^\beta \quad (4.2.1.4) \]

([Chi], equations (2.1), p.143, and (2.7), p.144). These polynomials form the monic OPS with respect to an MDF which we denote by \( \psi_{\alpha,\beta}^{(\alpha,\beta)} \). We define \( \psi_{\alpha,\beta}^{(\alpha,\beta)} \) by a weight function:
\[ \frac{d\psi_{\alpha,\beta}^{(\alpha,\beta)}}{dx} := \begin{cases} w(x), & \text{if } x \in (-1, 1) \\ 0, & \text{otherwise} \end{cases}, \quad (4.2.1.5) \]

where \( w(x) \) is given in (4.2.1.4); to be explicit, the orthogonality relation is
\[ \langle \hat{P}_m^{(\alpha,\beta)}, \hat{P}_n^{(\alpha,\beta)} \rangle_{\psi_{\alpha,\beta}^{(\alpha,\beta)}} \]
\[ = \int_{-1}^{1} \hat{P}_m^{(\alpha,\beta)}(x) \hat{P}_n^{(\alpha,\beta)}(x) (1 - x)^\alpha (1 + x)^\beta \, dx \]
\[ = 2^{2n+\alpha+\beta+1} \binom{2n + \alpha + \beta}{n}^{-1} B(n + \alpha + 1, n + \beta + 1) \delta_{mn} \quad (4.2.1.6) \]

(see the discussion in [Chi] beginning at the bottom of page 146 and ending on the next page) where \( B \) denotes the “Beta function” which can be given
in terms of the Gamma function by
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

The monic Jacobi polynomials can be specified in several alternate ways
to the Rodrigues’ type formula (4.2.1.1). For example, closely related to the
Rodrigues’ type formula is the system of differential equations
\[ \frac{d}{dx} \left( \rho(x)w(x) \frac{d}{dx} \hat{P}^{(\alpha,\beta)}_n(x) \right) + n(n + \alpha + \beta + 1)w(x)\hat{P}^{(\alpha,\beta)}_n(x) = 0, \]
\[ -1 < x < 1, \ n = 0, 1, 2, \ldots, \]
(4.2.1.7)

where \( \rho(x) \) is given by (4.2.1.3) and \( w(x) \) by (4.2.1.4) (see [Chi], eq. (2.19),
p. 148). The classical generating function is
\[ G(x, r) = \frac{2^{\alpha+\beta}}{R(1-r+R)^{\alpha}(1+r+R)^{\beta}} = \sum_{n=0}^{\infty} \binom{2n + \alpha + \beta}{n} 2^{-n} \hat{P}^{(\alpha,\beta)}_n(x) r^n \]
(4.2.1.8)

where
\[ R = R(x, r) = (1 - 2xr + r^2)^{1/2} \]
([Chi], eq. (2.32), p. 154, and eq. (2.7), p. 144). The fundamental recurrence
formula is
\[ \hat{P}^{(\alpha,\beta)}_n(x) = (x - c_n)\hat{P}^{(\alpha,\beta)}_{n-1}(x) - \lambda_n \hat{P}^{(\alpha,\beta)}_{n-2}(x), \ n = 1, 2, 3, \ldots, \]
(4.2.1.9)
where
\[
c_{n+1} = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}
\]
and
\[
\lambda_{n+1} = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)},
\]
except \(c_1 = (\beta - \alpha)/(\alpha + \beta + 2)\) when \(\alpha = -\beta\) ([Chi], eq. (2.29), p. 153).

Finally, and perhaps simplest of all, the monic Jacobi polynomials can be defined by the explicit formula
\[
\hat{P}_n^{(\alpha,\beta)}(x) = \binom{2n + \alpha + \beta}{n}^{-1} \sum_{k=0}^{n} \binom{n + \alpha}{n-k} \binom{n + \beta}{k} (x-1)^k (x+1)^{n-k},
\]
\(n = 0, 1, 2, \ldots\) (4.2.1.10)

([Chi], equations (2.6) and (2.7), p. 144).

To facilitate the reporting of the results of applying the Transformation Theorem and its consequences to the Jacobi class of systems of orthogonal polynomials, we find it useful to make the following definitions:

\[
a := v_+^{-1}(-1) = \frac{\lambda}{2} \left( -1 + \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right)
\]
and
\[
b := v_+^{-1}(1) = \frac{\lambda}{2} \left( 1 + \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right),
\]
where we continue to maintain the assumption made at the beginning of Chapter 2 that \(\lambda\) and \(\gamma\) are fixed positive real numbers. Thus, we have
assumed that $a$ and $b$ are the unique fixed positive real numbers having the
following properties:

$$-a = -v_+^{-1}(-1) = \frac{\lambda}{2} \left( 1 - \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right) = v_-^{-1}(1), \quad (4.2.1.13)$$

$$-b = -v_+^{-1}(1) = \frac{\lambda}{2} \left( -1 - \sqrt{1 + \frac{4\gamma}{\lambda^2}} \right) = v_-^{-1}(-1), \quad (4.2.1.14)$$

$$\lambda = b - a \quad (4.2.1.15)$$

and

$$\gamma = ab. \quad (4.2.1.16)$$

Using the Transformation Theorem, Theorem 3.2.1 and (4.2.1.5), we find
that $\tilde{\psi}_P^{(\alpha,\beta)}(x)$, given by

$$\frac{d\tilde{\psi}_P^{(\alpha,\beta)}}{dx} = \begin{cases} w(v(x)), & \text{if } v(x) \in (-1, 1) \\ 0, & \text{if } v(x) \in (-\infty, -1] \cup [1, \infty) \end{cases}, \quad (4.2.1.17)$$

is a strong moment distribution function for each choice of parameters $\alpha > -1$
and $\beta > -1$. Direct computation shows that

$$\frac{d\tilde{\psi}_P^{(\alpha,\beta)}}{dx} = \begin{cases} \frac{|x+a|^\alpha |b-x|^{\alpha} |x-a|^\beta |b+x|^{\beta}}{(b-a)^{\alpha+\beta} |x|^{\alpha+\beta}}, & \text{if } x \in (-b, -a) \cup (a, b) \\ 0, & \text{if } x \in (-\infty, -b] \cup [-a, 0) \cup (0, a) \cup [b, \infty) \end{cases}. \quad (4.2.1.18)$$

The monic OLPS with respect to $\tilde{\psi}_P^{(\alpha,\beta)}$ we denote by $\{\tilde{P}_m^{(\alpha,\beta)}(x)\}_{m=0}^\infty$.

By Theorem 2.3.1 (C), formula (4.2.1.10) and algebra, we find the explicit
formulas

\[ \widetilde{P}_{2n}^{(\alpha,\beta)}(x) = \frac{1}{(2n+\alpha+\beta)} x^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x+a)^k (x-b)^k (x-a)^{n-k} (x+b)^{n-k}, \]
\[ n = 0, 1, 2, \ldots, \quad (4.2.19) \]

and

\[ \widetilde{P}_{2n+1}^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2n+\alpha+\beta} x^{n+1} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x+a)^k (x-b)^k (x-a)^{n-k} (x+b)^{n-k}, \]
\[ n = 0, 1, 2, \ldots. \quad (4.2.20) \]

Considering Theorem 2.2.8 and equation (4.2.16), the orthogonality relation is

\[ \langle \widetilde{P}_{m}^{(\alpha,\beta)}, \widetilde{P}_{n}^{(\alpha,\beta)} \rangle = \int_{-a}^{-b} \tilde{P}_{m}^{(\alpha,\beta)}(x) \tilde{P}_{n}^{(\alpha,\beta)}(x) \frac{|x-a|^{\alpha}|b-x|^{\alpha}|x-a|^{\beta}|b+x|^{\beta}}{(b-a)^{\alpha+\beta}|x|^{\alpha+\beta}} \, dx \]
\[ + \int_{a}^{b} \tilde{P}_{m}^{(\alpha,\beta)}(x) \tilde{P}_{n}^{(\alpha,\beta)}(x) \frac{|x-a|^{\alpha}|b-x|^{\alpha}|x-a|^{\beta}|b+x|^{\beta}}{(b-a)^{\alpha+\beta}|x|^{\alpha+\beta}} \, dx \]
\[ = k_{m} \delta_{mn} \quad (4.2.21) \]

where

\[ k_{m} = \begin{cases} 
(b-a)^{2j+1}2^{2j+\alpha+\beta+1}(2j+\alpha+\beta) B(j+\alpha+1,j+\beta+1), & \text{if } m = 2j \\
(b-a)^{2j+1}2^{2j+\alpha+\beta+1}(2j+\alpha+\beta) B(j+\alpha+1,j+\beta+1), & \text{if } m = 2j+1 
\end{cases} \quad (4.2.22) \]

Considering Theorem 3.7.1 and equations (4.2.1) through (4.2.4), these
L-polynomials can also be defined by

$$\tilde{P}^{(\alpha, \beta)}_{2n}(x) = \frac{(b - a)^n}{K_n} \frac{d^n}{dv^n(x)} \left( \rho^n(v(x)) \ w(v(x)) \right), \ n = 0, 1, 2, \ldots,$$

(4.2.1.23)

and

$$\tilde{P}^{(\alpha, \beta)}_{2n+1}(x) = \frac{(a - b)^n}{a^n b^n K_n} \frac{d^n}{dv^n(x)} \left( \rho^n(v(x)) \ w(v(x)) \right), \ n = 0, 1, 2, \ldots,$$

(4.2.1.24)

where $K_n$ is given by (4.2.1.2), $\rho$ is given by (4.2.1.3), $w$ is given by (4.2.1.4) and $v(x) = \frac{1}{b - a} (x - ab/x)$. A similar result given by Theorem 3.8.1 and the system of differential equations (4.2.1.7) is that

$$\frac{d}{dx} \left( \frac{\rho(v(x)) w(v(x))}{v'(x)} \frac{d}{dx} \tilde{P}^{(\alpha, \beta)}_{2n}(x) \right) + n(n + \alpha + \beta + 1) v'(x) \tilde{P}^{(\alpha, \beta)}_{2n}(x) = 0,$$

$$a < |x| < b, \ n = 0, 1, 2, \ldots,$$

(4.2.1.25)

where $\rho$ is given by (4.2.1.3), $w$ is given by (4.2.1.4) and $v(x) = \frac{1}{b - a} (x - ab/x)$. By Theorem 3.9.1 and equation (4.2.1.8), these L-polynomials are also defined by the generating function

$$G \left( v(x), (b - a) r^2 \right) + \frac{r}{x} G \left( v(x), \frac{a - b}{ab} r^2 \right) = \sum_{m=0}^{\infty} b_m \tilde{P}^{(\alpha, \beta)}_m(x) r^m,$$

(4.2.1.26)

where $G$ is given by (4.2.1.8), $b_{2n+1} = b_{2n} = \binom{2n+\alpha+\beta}{n} 2^{-n}$ and $v(x) = \frac{1}{b - a} (x - ab/x)$. Lastly among our results to be mentioned in this section is the fundamental recurrence formula for the monic OLPS $\{\tilde{P}^{(\alpha, \beta)}_m(x)\}_{m=0}^{\infty}$. 

By Theorem 3.5.3 and the recurrence formula (4.2.1.9),
\[ \tilde{P}^{(\alpha, \beta)}_m(x) = \alpha_m \tilde{P}^{(\alpha, \beta)}_{m-1}(x) + (x(-1)^m + \beta_m) \tilde{P}^{(\alpha, \beta)}_{m-2}(x) \]
\[ + \gamma_m \tilde{P}^{(\alpha, \beta)}_{m-3}(x) + \delta_m \tilde{P}^{(\alpha, \beta)}_{m-4}(x), \quad m = 2, 3, 4, \ldots, \]
(4.2.1.27)
where we take \( \tilde{P}^{(\alpha, \beta)}_k(x) \equiv 0 \) for \( k < 0 \) and where \( \alpha_m, \beta_m, \gamma_m \) and \( \delta_m \) are given by, for \( n = 1, 2, 3, \ldots, \)
\[ \alpha_{2n} = (-ab)^n, \quad \alpha_{2n+1} = 0, \]
(4.2.1.28)
\[ \beta_{2n} = (a - b)c_n, \quad \beta_{2n+1} = \frac{b-a}{ab}c_n, \]
(4.2.1.29)
\[ \gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\frac{1}{ab})^n, \]
(4.2.1.30)
\[ \delta_{2n} = -(b-a)^2\lambda_n \text{ and } \delta_{2n+1} = -\left(\frac{b-a}{ab}\right)^2\lambda_n, \]
(4.2.1.31)
where
\[ c_{n+1} = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} \]
(4.2.1.32)
and
\[ \lambda_{n+1} = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}, \]
(4.2.1.33)
except
\[ c_1 = (\beta - \alpha)/(\alpha + \beta + 2) \text{ when } \alpha = -\beta. \]
(4.2.1.34)

4.2.2. The Gegenbauer Class. The monic Ultraspherical or Gegenbauer polynomials of parameter \( \alpha > -1, \alpha \neq -1/2 \), are denoted by \( \tilde{P}^{(\alpha)}_n(x) \) and can
be defined by

\[ \hat{P}_n^{(\alpha)}(x) = \hat{P}_n^{(\alpha,\alpha)}(x), \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (4.2.2.1)

For several reasons, including the fact that the symbol \( \lambda \) is already in use, we break with the tradition ([Chi], p.144, for example) of writing \( P_n^{(\lambda)}(x) \), \( \alpha = \lambda - 1/2 \neq -1/2 \), for the Gegenbauer polynomials. The class of Gegenbauer polynomials is a subclass of the Jacobi polynomials. Thus, for each choice of parameter \( \alpha > -1, \alpha \neq -1/2, \) \( \{\hat{P}_n^{(\alpha)}(x)\}_{n=0}^\infty \) is a monic OPS with respect to an MDF which we denote by \( \psi^{(\alpha)}_P \), where \( \psi^{(\alpha)}_P(x) = \psi^{(\alpha,\alpha)}_P(x) \). Considering (4.2.1.5) and (4.2.1.4), \( \psi^{(\alpha)}_P \) can be defined by the weight function

\[ w(x) = (1 - x^2)^\alpha, \]  \hspace{1cm} (4.2.2.2)

for \(-1 < x < 1\); specifically,

\[ \frac{d\psi^{(\alpha)}_P}{dx} = \begin{cases} 
(1 - x^2)^\alpha, & \text{if } x \in (-1,1) \\
0, & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (4.2.2.3)

The orthogonality relation is

\[ (\hat{P}_m^{(\alpha)}, \hat{P}_n^{(\alpha)})_{\psi^{(\alpha)}_P} = \int_{-1}^{1} \hat{P}_m^{(\alpha)}(x) \hat{P}_n^{(\alpha)}(x)(1 - x^2)^\alpha \, dx 
= 2^{2n+2\alpha+1} \binom{2n + 2\alpha}{n}^{-1} B(n + \alpha + 1, n + \alpha + 1) \delta_{mn}, \]  \hspace{1cm} (4.2.2.4)

by (4.2.1.6).

With the additional structure inherent in the Gegenbauer case comes similar simplifications as that demonstrated in (4.2.2.4) and computational
ease as we show next in the development of an explicit formula for the moments \( \mu_{\alpha}(\psi_P^{(\alpha)}) \). We first note that \( \mu_0(\psi_P^{(\alpha)}) = (\hat{P}_0^{(\alpha)}, \hat{P}_0^{(\alpha)})_{\psi_P^{(\alpha)}} \). Hence,

\[
\mu_0(\psi_P^{(\alpha)}) = 2^{2\alpha+1} B(\alpha + 1, \alpha + 1), \tag{4.2.5}
\]

by (4.2.4). Next, by symmetry considerations,

\[
\mu_{2k+1}(\psi_P^{(\alpha)}) = \int_{-1}^{1} x^{2k+1} (1 - x^2)^{\alpha} \, dx = 0, \; k = 0, 1, 2, \ldots. \tag{4.2.6}
\]

Integration by parts gives

\[
\mu_{2k}(\psi_P^{(\alpha)}) = \int_{-1}^{1} x^{2k} (1 - x^2)^{\alpha} \, dx
\]

\[
= \frac{2k - 1}{2(\alpha + 1)} \int_{-1}^{1} x^{2k-2} (1 - x^2)^{\alpha+1} \, dx
\]

\[
= \frac{2k - 1}{2(\alpha + 1)} \int_{-1}^{1} (x^{2k-2} - x^{2k}) (1 - x^2)^{\alpha} \, dx
\]

\[
= \frac{2k - 1}{2(\alpha + 1)} \left( \mu_{2k-2}(\psi_P^{(\alpha)}) - \mu_{2k}(\psi_P^{(\alpha)}) \right).
\]

Hence,

\[
\mu_{2k}(\psi_P^{(\alpha)}) = \frac{2k - 1}{2\alpha + 2k + 1} \mu_{2k-2}(\psi_P^{(\alpha)}) = \mu_0(\psi_P^{(\alpha)}) \prod_{j=1}^{k} \frac{2(k-j)+1}{2(\alpha + k-j) + 3},
\]

\[ k = 1, 2, 3, \ldots. \tag{4.2.7} \]

By (4.2.1.1), the monic Gegenbauer polynomials have the explicit representation given by (4.2.1.10) with \(-1/2 \neq \beta = \alpha > -1:\)

\[
\hat{P}_n^{(\alpha)}(x) = \left( \frac{2n + 2\alpha}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \alpha}{k} (x - 1)^k (x + 1)^{n-k},
\]

\[ n = 0, 1, 2, \ldots. \tag{4.2.8} \]

These polynomials can also be defined by the generating function (4.2.1.8),
the system of differential equations (4.2.1.7), the Rodrigues’ type formula
(4.2.1.1), or the fundamental recurrence (4.2.1.9). We should note that sim-
pler generating functions are known for these polynomials ([Chi], eq. (2.33),
p. 154, for example). It is also noteworthy that the fundamental recurrence
formula algebraically simplifies significantly:

\[ \hat{P}_n^{(\alpha)}(x) = x\hat{P}_{n-1}^{(\alpha)}(x) - \frac{(n-1)(n-1+2\alpha)}{(2n+2\alpha-2)^2-1}\hat{P}_{n-2}^{(\alpha)}(x), \quad n = 1, 2, 3, \ldots. \]

(4.2.2.9)

The L-polynomials which result in this case have the explicit formulas
given in equations (4.2.1.19) and (4.2.1.20), where \(-1/2 \neq \beta = \alpha > -1\):

\[ \tilde{P}_2^{(\alpha)}(x) = \]

\[ \frac{1}{(2n+2\alpha)_n} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\alpha}{k} (x+a)^k(x-b)^k(x-a)^{n-k}(x+b)^{n-k}, \]

\[ n = 0, 1, 2, \ldots, \]

(4.2.2.10)

and

\[ \tilde{P}_{2n+1}^{(\alpha)}(x) = \]

\[ \frac{(-1)^n}{(2n+2\alpha)_n} a^n b^n x^{n+1} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\alpha}{k} (x+a)^k(x-b)^k(x-a)^{n-k}(x+b)^{n-k}, \]

\[ n = 0, 1, 2, \ldots, \]

(4.2.2.11)

\{\tilde{P}_m^{(\alpha)}\}_{m=0}^{\infty} is the monic OLPS with respect to the SMDF \tilde{\psi}_P^{(\alpha)} where, by
(4.2.1.18),

\[ \frac{d\tilde{\psi}_P^{(\alpha)}}{dx} = \begin{cases} \frac{(x^2-a^2)^n(b^2-x^2)^\alpha}{(b-a)^{2\alpha}[x]^{\alpha}}, & \text{if } x \in (-b, -a) \cup (a, b) \\
0, & \text{if } x \in (-\infty, -b] \cup [-a, 0) \cup (0, a] \cup [b, \infty) \end{cases}, \]

(4.2.2.12)
Considering (4.2.1.21) and (4.2.1.22), the orthogonality relation is

\[
(\tilde{P}_m^{(\alpha)}, \tilde{P}_n^{(\alpha)})_P^{(\alpha)} = \int_{-b}^{-a} \tilde{P}_m^{(\alpha)}(x) \tilde{P}_n^{(\alpha)}(x) \frac{(x^2 - a^2)^{\alpha}(b^2 - x^2)^{\alpha}}{(b - a)^{2\alpha}|x|^{2\alpha}} \, dx \\
+ \int_{a}^{b} \tilde{P}_m^{(\alpha)}(x) \tilde{P}_n^{(\alpha)}(x) \frac{(x^2 - a^2)^{\alpha}(b^2 - x^2)^{\alpha}}{(b - a)^{2\alpha}|x|^{2\alpha}} \, dx
\]

\[= k_m \delta_{mn} \quad (4.2.2.13)\]

where

\[
k_m = \begin{cases} 
(b - a)^{2j + 2j + 2\alpha + 1} \binom{2j + 2\alpha}{j}^{-1} B(j + \alpha + 1, j + \alpha + 1), & \text{if } m = 2j \\
(b - a)^{2j + 1} \binom{2j + 2\alpha}{j}^{-1} B(j + \alpha + 1, j + \alpha + 1), & \text{if } m = 2j + 1.
\end{cases} \quad (4.2.2.14)
\]

We can find expressions for the moments of \(\tilde{\psi}_P^{(\alpha)}\) by considering Theorem 2.2.5, Theorem 3.3.5 and equations (4.2.2.5), (4.2.2.6) and (4.2.2.7):

\[
\mu_{2l+1}(\tilde{\psi}_P^{(\alpha)}) = 0, \quad l = 0, \pm 1, \pm 2, \ldots, \quad (4.2.2.15)
\]

\[
\mu_{-2l-2}(\tilde{\psi}_P^{(\alpha)}) = (ab)^{-2l-1} \mu_{2l}(\tilde{\psi}_P^{(\alpha)}), \quad l = 0, 1, 2, \ldots, \quad (4.2.2.16)
\]

and

\[
\mu_{2l}(\tilde{\psi}_P^{(\alpha)}) = \sum_{i=0}^{l} \binom{2l - i}{i} (ab)^i (b - a)^{2l - 2i + 1} \mu_{2(l-i)}(\psi_P^{(\alpha)}), \quad l = 0, 1, 2, \ldots, \quad (4.2.2.17)
\]

where \(\mu_{2(l-i)}(\psi_P^{(\alpha)})\) is given by (4.2.2.5) and (4.2.2.7) as

\[
\mu_{2(l-i)}(\psi_P^{(\alpha)}) = 2^{2\alpha + 1} B(\alpha + 1, \alpha + 1) \prod_{j=1}^{l-i} \frac{2(l - i - j) + 1}{2(\alpha + l - i - j) + 3}. \quad (4.2.2.18)
\]
By taking $-1/2 \neq \beta = \alpha > -1$ in each formula, the monic OLPS $\{\tilde{P}_m^{(\alpha)}\}_{m=0}^{\infty}$ can also be defined by the Rodrigues’ type formulas (4.2.1.23) and (4.2.1.24), by the system of differential equations (4.2.1.25), by the generating function (4.2.1.26), or by the recurrence formula (4.2.1.27). For example, the fundamental recurrence formula is simply

$$\tilde{P}_m^{(\alpha)}(x) = \alpha_m \tilde{P}_{m-1}^{(\alpha)}(x) + x(-1)^m \tilde{P}_{m-2}^{(\alpha)}(x) + \gamma_m \tilde{P}_{m-3}^{(\alpha)}(x) + \delta_m \tilde{P}_{m-4}^{(\alpha)}(x),$$

$$m = 2, 3, 4, \ldots, \quad (4.2.2.19)$$

where we set $\tilde{P}_k^{(\alpha)}(x) \equiv 0$ for $k < 0$ and where $\alpha_m$, $\gamma_m$ and $\delta_m$ are given by

$$\alpha_{2n} = (-ab)^n, \quad \alpha_{2n+1} = 0, \quad (4.2.2.20)$$

$$\gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\frac{1}{ab})^n, \quad (4.2.2.21)$$

$$\delta_{2n} = -(b-a)^2 \frac{(n-1)(n-1+2\alpha)}{(2n+2\alpha-2)^2-1} \text{ and } \delta_{2n+1} = -(\frac{b-a}{ab})^2 \frac{(n-1)(n-1+2\alpha)}{(2n+2\alpha-2)^2-1}, \quad (4.2.2.22)$$

for $n = 1, 2, 3, \ldots$.

4.2.3. The Tchebycheff Polynomials of the First Kind. Some of the formulas involving both $n$ and $\alpha$ for the monic Gegenbauer polynomials $\tilde{P}_n^{(\alpha)}(x)$ discussed in the previous section do not hold in a strict sense for $n = 2$ and $\alpha = -1/2$, yet can all be extended by considering the limit as $\alpha$ tends to $-1/2$. For this reason, the monic Tchebycheff polynomials of the first kind, which are denoted by $\tilde{T}_n(x)$, are sometimes considered an instance of the
class of Gegenbauer polynomials. These polynomials can be defined by
\[
\hat{T}_n(x) = \hat{P}_n^{(-1/2,-1/2)}(x), \quad n = 0, 1, 2, \ldots, \quad (4.2.3.1)
\]
hence the Tchebycheff polynomials of the first kind are an example from the class of Jacobi polynomials with \(\alpha = \beta = -1/2\).

\(\{T_n(x)\}_{n=0}^{\infty}\) is the monic OPS with respect to the MDF \(\psi_T\) given by
\[
\frac{d\psi_T}{dx} = \begin{cases} 
\frac{1}{\sqrt{1-x^2}}, & \text{if } x \in (-1,1) \\
0, & \text{otherwise}
\end{cases} \quad (4.2.3.2)
\]
To be specific, by (4.2.1.6), noting that
\[
B(n + 1/2, n + 1/2) = \frac{\pi}{16^n} \binom{2n}{n},
\]
the orthogonality relation is
\[
(\hat{T}_m, \hat{T}_n)_{\psi_T} = \int_{-1}^{1} \hat{T}_m(x)\hat{T}_n(x) \frac{1}{\sqrt{1-x^2}} \, dx \\
= 4^n \binom{2n-1}{n}^{-1} B(n + 1/2, n + 1/2) \delta_{mn} \quad (4.2.3.3)
\]
where
\[
k_n = \begin{cases} 
\pi, & \text{if } n = 0 \\
2^{1-2n} \pi, & \text{if } n \geq 1
\end{cases}
\]
Considering (4.2.2.5), (4.2.2.6) and (4.2.2.7) with \(\alpha = -1/2\), the moments satisfy
\[
\mu_{2k+1}(\psi_T) = 0, \quad k = 0, 1, 2, \ldots, \quad (4.2.3.4)
\]
and
\[
\mu_{2k}(\psi) = \pi \prod_{j=1}^{k} \frac{2(k-j)+1}{2(k-j+1)} = \frac{(2k)!}{4^k (k!)^2} = \frac{\pi}{4^k} \binom{2k}{k}, \quad k = 0, 1, 2, \ldots.
\]

(4.2.3.5)

The monic Tchebycheff polynomials of the first kind are sometimes defined by the formula

\[
\hat{T}_n(x) = h_n \cos(n \cos^{-1} x), \quad -1 \leq x \leq 1, \quad n = 0, 1, 2, \ldots
\]

(4.2.3.6)

where

\[
h_n = \begin{cases} 
1, & \text{if } n = 0 \\
2^{1-n}, & \text{if } n \geq 1 
\end{cases}
\]

([Chi], eq. (1.4), p. 1). From the representation (4.2.3.6), it is easy to see that the zeros of \(\hat{T}_n(x)\), which we write as \(x_{nj}\), are given by

\[
x_{nj} = \cos \left( \pi - \frac{2j-1}{2n} \pi \right), \quad j = 1, 2, 3, \ldots, n,
\]

(4.2.3.7)

and have the ordering

\[
x_{n1} < x_{n2} < x_{n3} < \cdots < x_{nn}.
\]

(4.2.3.8)

By (4.2.3.1) and (4.2.1.10), these polynomials also have the explicit representation

\[
\hat{T}_n(x) = \binom{2n-1}{n}^{-1} \sum_{k=0}^{n} \binom{n-1/2}{n-k} \binom{n-1/2}{k} (x-1)^k(x+1)^{n-k},
\]

\[
n = 0, 1, 2, \ldots.
\]

(4.2.3.9)
Similarly, these polynomials are given by the Rodrigues’ type formula (4.2.1.1),
the system of differential equations (4.2.1.7), the classical generating function
(4.2.1.8), or the fundamental recurrence formula (4.2.1.9) by taking \( \beta = \alpha \to -1/2 \). For example, the fundamental recurrence formula is

\[
\tilde{T}_n(x) = x\tilde{T}_{n-1}(x) - \frac{1}{4}\tilde{T}_{n-2}(x), \quad n = 1, 2, 3, \ldots, \tag{4.2.3.10}
\]

except

\[
\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x). \tag{4.2.3.11}
\]

There is also a simpler generating function, compared to the one given by
(4.2.1.8):

\[
\frac{1 - x^r}{1 - 2x^r + r^2} = \sum_{n=0}^{\infty} a_n \tilde{T}_n(x) \ r^n, \tag{4.2.3.12}
\]

where

\[
a_n = \begin{cases} 
1, & \text{if } n = 0 \\
2^{n-1}, & \text{if } n \geq 1
\end{cases}
\]

(see [Chi], eq. (2.37), p. 155).

Applying the Transformation Theorem with \( \lambda = b - a \) and \( \gamma = ab \) to
the Tchebycheff polynomials of the first kind results in a monic orthogonal
Laurent polynomial sequence \( \{\tilde{T}_m(x)\}_{m=0}^{\infty} \) which, by (4.2.3.12) and Theorem
3.9.1, can be defined by the generating function
\[
\frac{1 - (x - \frac{ab}{x})r^2}{1 - 2(x - \frac{ab}{x})r^2 + (b - a)^2r^4} + \frac{r}{x} \frac{1 + (\frac{x}{ab} - \frac{1}{x})r^2}{1 - 2(\frac{x}{ab} - \frac{1}{x})r^2 + \frac{(b-a)^2}{a^2b^2}r^4} = \sum_{m=0}^{\infty} b_m \tilde{T}_m(x) r^m,
\]

(4.2.3.13)

where, for \( n = 0, 1, 2, \ldots \),

\[
b_{2n+1} = b_{2n} = \begin{cases} 
1, & \text{if } n = 0 \\
2^{n-1}, & \text{if } n \geq 1.
\end{cases}
\]

(4.2.3.14)

By (4.2.3.1),

\[
\tilde{T}_m(x) = \tilde{P}_m^{(-1/2,-1/2)}(x), \ m = 0, 1, 2, \ldots
\]

(4.2.3.15)

Thus, by (4.2.1.18), these L-polynomials are orthogonal with respect to the SMDF, which we denote by \( \tilde{\psi}_T \), given by

\[
\frac{d \tilde{\psi}_T}{dx} = \begin{cases} 
\frac{(b-a) |x|}{\sqrt{b^2-x^2}\sqrt{x^2-a^2}}, & \text{if } x \in (-b, -a) \cup (a, b) \\
0, & \text{if } x \in (-\infty, -b] \cup [-a, 0) \cup (0, a] \cup [b, \infty).
\end{cases}
\]

(4.2.3.16)

Considering (4.2.3.3) and Theorem 2.2.8 for example, the orthogonality relation is

\[
(\tilde{T}_m, \tilde{T}_n)_{\tilde{\psi}_T} = \int_{-b}^{-a} \tilde{T}_m(x)\tilde{T}_n(x) \frac{(b-a) |x|}{\sqrt{b^2-x^2}\sqrt{x^2-a^2}} \, dx \\
+ \int_{a}^{b} \tilde{T}_m(x)\tilde{T}_n(x) \frac{(b-a) |x|}{\sqrt{b^2-x^2}\sqrt{x^2-a^2}} \, dx
\]

(4.2.3.17)

\[
= k_m \delta_{mn}
\]
where

\[ k_m = \begin{cases} 
  (b - a)\pi, & \text{if } m = 0 \\
  (b - a)^{2j+1} 2^{1-2j} \pi, & \text{if } m = 2j \geq 2 \\
  (b - a)^{2j+1} 2^{1-2j} \pi, & \text{if } m = 2j + 1 
\end{cases} \]  \hspace{1cm} (4.2.3.18)

We can find expressions for the moments of \( \tilde{\psi}_T \) by considering Theorem 2.2.5 (B), Theorem 3.3.5 and equations (4.2.3.4) and (4.2.3.5):

\[ \mu_{2l+1}(\tilde{\psi}_T) = 0, \quad l = 0, \pm 1, \pm 2, \ldots, \]  \hspace{1cm} (4.2.3.19)

\[ \mu_{-2l-2}(\tilde{\psi}_T) = (ab)^{-2l-1} \mu_{2l}(\tilde{\psi}_T), \quad l = 0, 1, 2, \ldots, \]  \hspace{1cm} (4.2.3.20)

and

\[ \mu_{2l}(\tilde{\psi}_T) = \sum_{i=0}^{l} \binom{2l - i}{i} (ab)^i (b - a)^{2l-2i+1} \frac{(2l - 2i)! \pi}{4^{l-i}((l - i)!)^2} \]

\[ = \frac{\pi}{4^l} \sum_{i=0}^{l} \binom{2l - i}{i} \binom{2l - 2i}{l - i} (4ab)^i (b - a)^{2l-2i+1}, \quad l = 0, 1, 2, \ldots. \]  \hspace{1cm} (4.2.3.21)

By taking \( \beta = \alpha \) and letting \( \alpha \) tend to \(-1/2\) in the expressions obtained for the Laurent polynomials in the general Jacobi case, we can find several formulas for the L-polynomials of the OLPS \( \{\tilde{T}_m(x)\}_{m=0}^\infty \) in addition to the generating function (4.2.3.13). In particular, by (4.2.1.19) and (4.2.1.20),

\[ \tilde{T}_{2n}(x) = \]

\[ \frac{1}{(2n-1)_n} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n-1/2}{n-k} \binom{n-1/2}{k} (x + a)^k (x - b)^k (x - a)^{n-k} (x + b)^{n-k}, \quad n = 0, 1, 2, \ldots, \]  \hspace{1cm} (4.2.3.22)
\[ \tilde{T}_{2n+1}(x) = \frac{(-1)^n}{(2n-1)/n} \frac{1}{a^n b^n x^{n+1}} \sum_{k=0}^{n} \binom{n-1/2}{n-k} \binom{n-1/2}{k} (x + a)^k (x - b)^k (x - a)^{n-k} (x + b)^{n-k}, \]
\[ n = 0, 1, 2, \ldots, \quad (4.2.3.23) \]

and, by equations (4.2.1.27) through (4.2.1.34),
\[ \tilde{T}_m(x) = \alpha_m \tilde{T}_{m-1}(x) + x(-1)^m \tilde{T}_{m-2}(x) + \gamma_m \tilde{T}_{m-3}(x) + \delta_m \tilde{T}_{m-4}(x), \]
\[ m = 2, 3, 4, \ldots, \quad (4.2.3.24) \]

where we take \( \tilde{T}_k(x) \equiv 0 \) for \( k < 0 \) and where \( \alpha_m, \gamma_m \) and \( \delta_m \) are given by
\[ \alpha_{2n} = (-ab)^n, \quad \alpha_{2n+1} = 0, \quad (4.2.3.25) \]
\[ \gamma_{2n} = 0, \quad \gamma_{2n+1} = (-1/ab)^n, \quad (4.2.3.26) \]
\[ \delta_{2n} = -\frac{1}{4}(b-a)^2 \text{ and } \delta_{2n+1} = -\frac{1}{4}\left(\frac{b-a}{ab}\right)^2, \quad (4.2.3.27) \]

for \( n = 1, 2, 3, \ldots, \) except
\[ \delta_4 = -\frac{1}{2}(b-a)^2 \text{ and } \delta_5 = -\frac{1}{2}\left(\frac{b-a}{ab}\right)^2. \quad (4.2.3.28) \]

Lastly in this section, we mention that the zeros of \( \tilde{T}_m(x) \), for each \( m \geq 2 \), can be found by considering Theorem 3.4.1 and equations (4.2.3.7) and (4.2.3.8). The zeros, which we denote by \( x_{n,j}^{\pm} \), of \( \tilde{T}_{2n}(x) \) and \( \tilde{T}_{2n+1}(x) = (\frac{-1}{ab})^n \frac{1}{x} \tilde{T}_{2n}(x) \), for \( n \geq 1 \), are given by
\[ x_{n,j}^{\pm} = \frac{b-a}{2} \left( \cos \left( \pi - \frac{2j - 1}{2n} \pi \right) \pm \sqrt{\cos^2 \left( \pi - \frac{2j - 1}{2n} \pi \right) + \frac{4ab}{(b-a)^2}} \right), \]
\[ j = 1, 2, 3, \ldots, n, \quad (4.2.3.29) \]
and have the ordering

\[ x_{n1}^- < x_{n2}^- < \cdots < x_{nn}^- < 0 < x_{n1}^+ < x_{n2}^+ < \cdots < x_{nn}^+. \] (4.2.3.30)

4.2.4. The Legendre Polynomials. The monic Legendre polynomials we denote by \( \hat{P}_n(x) \), \( n = 0, 1, 2, \ldots \). These polynomials can be defined by

\[ \hat{P}_n(x) = \hat{P}_n^{(0)}(x) = \hat{P}_n^{(0,0)}(x), \quad n = 0, 1, 2, \ldots, \] (4.2.4.1)

hence they are an instance of the Gegenbauer and Jacobi classes of polynomials. The monic Legendre polynomials thus form an OPS with respect to an MDF, which we denote by \( \psi_P \), given by

\[ \frac{d\psi_P}{dx} = \begin{cases} 1 & \text{if } x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}, \] (4.2.4.2)

according to (4.2.2.3). Considering (4.2.4.1) and (4.2.2.4), the orthogonality relation is

\[ (\hat{P}_m, \hat{P}_n)_{\psi_P} = \int_{-1}^{1} \hat{P}_m(x)\hat{P}_n(x) \, dx = \frac{2^{2n+1}(n!)^4}{(2n)!(2n+1)!} \delta_{mn}. \] (4.2.4.3)

The moments are

\[ \mu_{2k+1}(\psi_P) = 0, \quad k = 0, 1, 2, \ldots, \] (4.2.4.4)
and
\[ \mu_{2k}(\psi_P) = \frac{2}{2k+1}, \quad k = 0, 1, 2, \ldots, \] (4.2.4.5)
by (4.2.2.5), (4.2.2.6) and (4.2.2.7).

As in the general case of the Jacobi polynomials, the monic Legendre polynomials can be specified in many ways. By (4.2.4.1) and (4.2.1.1), they are given by the Rodrigues’ formula
\[ \hat{P}_n(x) = \frac{(-1)^n n!}{(2n)!} \frac{d^n}{dx^n}((1-x^2)^n), \quad n = 0, 1, 2, \ldots. \] (4.2.4.6)

We can use (4.2.4.6), the Binomial Theorem and linearity of the derivative to find another expression for these polynomials:
\begin{align*}
\hat{P}_n(x) &= \frac{(-1)^n n!}{(2n)!} \frac{d^n}{dx^n}((1-x^2)^n) \\
&= \frac{(-1)^n n!}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^k}{dx^k} x^{2k} \\
&= \sum_{j=0}^{n} \hat{P}_{n,j} x^j, \quad n = 0, 1, 2, \ldots,
\end{align*}
(4.2.4.7)
where
\[ \hat{P}_{n,j} := \begin{cases} (-1)^{\frac{n-j}{2}} \frac{n!}{(\frac{n+j}{2})!} (\frac{n}{2})_n (j+1)_n, & \text{if } j + n \text{ is even} \\
0, & \text{otherwise} \end{cases}. \] (4.2.4.8)
Here, we have used “Pochhammer’s symbol” \((z)_n\) defined by \((z)_0 := 1\), and
\[ (z)_n := z(z+1)(z+2) \cdots (z+n-1), \quad n = 1, 2, 3, \ldots. \]
Closely related to (4.2.4.6) is the system of differential equations, given by
(4.2.1.7) as
\[
\frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \hat{P}_n(x) \right) + n(n + 1) \hat{P}_n(x) = 0, \quad -1 < x < 1, \quad n = 0, 1, 2, \ldots .
\] (4.2.4.9)

There is a relatively simple generating function for the monic Legendre polynomials:
\[
(1 - 2x r + r^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!^2} \hat{P}_n(x) r^n
\] (2.4.10)

([Chi], eq. (2.34), p. 154, and eq. (2.7), P. 144). Considering (4.2.4.1) and (4.2.2.9), the fundamental recurrence formula is
\[
\hat{P}_n(x) = x \hat{P}_{n-1}(x) - \frac{(n-1)^2}{(2n-3)(2n-1)} \hat{P}_{n-2}(x), \quad n = 1, 2, 3, \ldots .
\] (4.2.4.11)

Lastly, these polynomials are also given by
\[
\hat{P}_n(x) = \frac{(n!)^2}{(2n)!} \sum_{k=0}^{n} \binom{n}{n-k} \binom{n}{k} (x-1)^k (x+1)^{n-k}, \quad n = 0, 1, 2, \ldots
\] (4.2.4.12)

according to (4.2.4.1) and (4.2.2.8).

Applying the Transformation Theorem, with \( \lambda = b - a \) and \( \gamma = ab \), to the system of monic Legendre polynomials results in a system of orthogonal Laurent polynomials. We denote the L-polynomials by \( \tilde{P}_m(x) \). By (4.2.4.1),
\[
\tilde{P}_m(x) = P_m^{(0)}(x) = \tilde{P}_m^{(0,0)}(x), \quad m = 0, 1, 2, \ldots
\] (4.2.4.13)
We denote the resulting SMDF by $\tilde{\psi}_P$, where

$$\frac{d\tilde{\psi}_P}{dx} = \begin{cases} 1, & \text{if } x \in (-b, -a) \cup (a, b) \\ 0, & \text{if } x \in (-\infty, -b] \cup [-a, 0) \cup (0, a] \cup [b, \infty). \end{cases} \quad (4.2.4.14)$$

considering (4.2.4.13) and (4.2.2.3). By Theorem 2.2.8 and equation (4.2.4.3), we see that the orthogonality relation is

$$\langle \tilde{P}_m, \tilde{P}_n \rangle_{\tilde{\psi}_P} = \int_{-b}^{-a} \tilde{P}_m(x) \tilde{P}_n(x) \, dx + \int_{a}^{b} \tilde{P}_m(x) \tilde{P}_n(x) \, dx = k_m \delta_{mn} \quad (4.2.4.15)$$

where

$$k_m = \begin{cases} (b - a)^{2j+1} \frac{2^{2j+1}(j)!^4}{(2j)!^2(2j+1)!^2}, & \text{if } m = 2j \\ \left(\frac{b-a}{ab}\right)^{2j+1} \frac{2^{2j+1}(j)!^4}{(2j)!^2(2j+1)!^2}, & \text{if } m = 2j + 1. \end{cases} \quad (4.2.4.16)$$

It is a simple matter to calculate the moments for $\tilde{\psi}_P$:

$$\mu_{2l+1}(\tilde{\psi}_P) = 0, \quad l = 0, \pm 1, \pm 2, \ldots, \quad (4.2.4.17)$$

and

$$\mu_{2l}(\tilde{\psi}_P) = \int_{-b}^{-a} x^{2l} \, dx + \int_{a}^{b} x^{2l} \, dx = 2 \int_{a}^{b} x^{2l} \, dx = 2 \frac{b^{2l+1} - a^{2l+1}}{2l+1}, \quad l = 0, 1, 2, \ldots. \quad (4.2.4.18)$$

$$\left(1 - 2 \left(\frac{x}{a} - \frac{ab}{x}\right) r^2 + (b - a)^2 r^4\right)^{-1/2} + \frac{r}{x} \left(1 + 2 \left(\frac{x}{ab} - \frac{1}{x}\right) r^2 + \left(\frac{b-a}{ab}\right)^2 r^4\right)^{-1/2} = \sum_{m=0}^{\infty} b_m \tilde{P}_m(x) r^m, \quad (4.2.4.19)$$
where
\[ b_{2n+1} = b_2 n = \frac{(2n)!}{2^n (n!)^2}, \quad n = 0, 1, 2, \ldots, \quad (4.2.20) \]
is a generating function for the OLPS \( \{ \tilde{P}_m(x) \}_{m=0}^{\infty} \), by (4.2.10) and Theorem 3.9.1.

Among the recurrence relations provided in section 3.5 for transformed OPS’s is the fundamental recurrence formula. Considering Theorem 3.5.3 and equations (4.2.4.11), the fundamental recurrence formula for \( \{ \tilde{P}_m(x) \}_{m=0}^{\infty} \) is
\[
\tilde{P}_m(x) = \alpha_m \tilde{P}_{m-1}(x) + x(1)^m \tilde{P}_{m-2}(x) + \gamma_m \tilde{P}_{m-3}(x) + \delta_m \tilde{P}_{m-4}(x),
\]
\[ m = 2, 3, 4, \ldots, \quad (4.2.4.21) \]
where we take \( \tilde{P}_k(x) \equiv 0 \) for \( k < 0 \) and where \( \alpha_m, \gamma_m \) and \( \delta_m \) are given by
\[
\begin{align*}
\alpha_{2n} &= (-ab)^n, & \alpha_{2n+1} &= 0, & (4.2.4.22) \\
\gamma_{2n} &= 0, & \gamma_{2n+1} &= (-\frac{1}{ab})^n, & (4.2.4.23) \\
\delta_{2n} &= -(b-a)^2 \frac{(n-1)^2}{(2n-3)(2n-1)} & \text{and} & \delta_{2n+1} &= -(\frac{b-a}{ab})^2 \frac{(n-1)^2}{(2n-3)(2n-1)}, & (4.2.4.24)
\end{align*}
\]
for \( n = 1, 2, 3, \ldots \). From the explicit formulas (4.2.2.10) and (4.2.2.11) for the monic Gegenbauer polynomials, with \( \alpha = 0 \), we also have
\[
\tilde{P}_{2n}(x) = \frac{(n!)^2}{(2n+1)!} x^n \sum_{k=0}^{n} \binom{n}{n-k} \binom{n}{k} (x+a)^k (x-b)^k (x-a)^{n-k} (x+b)^{n-k},
\]
\[ n = 0, 1, 2, \ldots, \quad (4.2.4.25) \]
and
\[
\tilde{P}_{2n+1}(x) = \frac{(-1)^n (n!)^2}{(2n+1)! a^n b^n x^{n+1}} \sum_{k=0}^{n} \binom{n}{n-k} \binom{n}{k} (x+a)^k (x-b)^k (x-a)^{n-k} (x+b)^{n-k},
\]
\[n = 0, 1, 2, \ldots. \quad (4.2.4.26)\]

The translations (4.2.1.23) and (4.2.1.24) of the Rodrigues’ formula \((\alpha = \beta = 0)\) yield
\[
\tilde{P}_{2n}(x) = \frac{(a-b)^n n!}{(2n)!} \frac{d^n}{dv^n(x)} \left( (1-v^2(x))^n \right), \quad n = 0, 1, 2, \ldots, \quad (4.2.4.27)
\]
and
\[
\tilde{P}_{2n+1}(x) = \frac{(b-a)^n n!}{a^n b^n (2n)!} \frac{d^n}{dv^n(x)} \left( (1-v^2(x))^n \right), \quad n = 0, 1, 2, \ldots, \quad (4.2.4.28)
\]
where \(v(x) = \frac{1}{b-a}(x - ab/x)\). The system of differential equation (4.2.1.25) reduces in this case to
\[
\frac{d}{dx} \left( \frac{(b^2 - x^2)(x^2 - a^2)}{x^2 + ab} \frac{d}{dx} \tilde{P}_{2n}(x) \right) + n(n+1) \frac{x^2 + ab}{x^2} \tilde{P}_{2n}(x) = 0,
\]
\[a < |x| < b, \quad n = 0, 1, 2, \ldots, \quad (4.2.4.29)\]
Lastly, considering Theorem 3.6.1 and the representation (4.2.4.7) involving the polynomial coefficients (4.2.4.8), we can find explicit formulas for the L-polynomial coefficients:
\[
\tilde{P}_{2n}(x) = \sum_{k=-n}^{n} \tilde{P}_{2n,k} \ x^j, \quad n = 0, 1, 2, \ldots, \quad (4.2.4.30)
\]
\[
\tilde{P}_{2n+1}(x) = \sum_{k=-n-1}^{n-1} (ab)^{-n} \tilde{P}_{2n,k+1} x^j, \quad n = 0, 1, 2, \ldots, \quad (4.2.4.31)
\]

where

\[
\tilde{P}_{2n,k} = \sum_{0 \leq i \leq j \leq n, 2i - j = k} \binom{j}{i} (b - a)^{n-j} (-ab)^{j-i} \tilde{P}_{n,j}, \quad (4.2.4.32)
\]

and where \( \tilde{P}_{n,j} \) is given by (4.2.4.8) as

\[
\tilde{P}_{n,j} := \begin{cases} 
( -1 )^{ \frac{n-j}{2} } \frac{n!}{(2n)!} \frac{n}{(n+j)} (j + 1)_n, & \text{if } j + n \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

4.3. The Generalized Hermite Class

4.3.1. The General Class. We denote the monic generalized Hermite polynomials of parameter \( \alpha > -1/2 \) by \( \hat{H}_n^{(\alpha)}(x) \), \( n = 0, 1, 2, \ldots \).

\[
\hat{H}_n^{(\alpha)}(x) = (-1)^k k! \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k-j} \frac{(-1)^j}{j!} x^{2j}, \quad k = 0, 1, 2, \ldots, \quad (4.3.1.1)
\]

and

\[
\hat{H}_{2k+1}^{(\alpha)}(x) = (-1)^k k! \sum_{j=0}^{k} \binom{k + \alpha + 1/2}{k-j} \frac{(-1)^j}{j!} x^{2j+1}, \quad k = 0, 1, 2, \ldots, \quad (4.3.1.2)
\]

(see [Chi], (2.43), p. 156, and (2.11), p. 145). \( \{ \hat{H}_n^{(\alpha)}(x) \}_{n=0}^{\infty} \) is the monic
OPS with respect to the MDF $\psi^{(\alpha)}_H$ which is given by

\[
\frac{d\psi^{(\alpha)}_H}{dx} = |x|^{2\alpha} e^{-x^2}, \ x \in \mathbb{R}.
\] (4.3.1.3)

The orthogonality relation is

\[
(\hat{H}_m^{(\alpha)}, \hat{H}_n^{(\alpha)})_{\psi^{(\alpha)}_H} = \int_{-\infty}^{\infty} \hat{H}_m^{(\alpha)}(x)\hat{H}_n^{(\alpha)}(x)|x|^{2\alpha}e^{-x^2} \, dx = \left[\frac{n}{2}\right]! \Gamma\left(\left[\frac{n+1}{2}\right] + \alpha + \frac{1}{2}\right) \delta_{mn}
\] (4.3.1.4)

([Chi], eq. (2.45), p. 157), where $[z]$ denotes the integer part of $z$.

We can easily find expressions for the moments of $\psi^{(\alpha)}_H$. By symmetry properties,

\[
\mu_{2k+1}(\psi^{(\alpha)}_H) = 0, \ k = 0, 1, 2, \ldots.
\] (4.3.1.5)

Since $\mu_0(\psi^{(\alpha)}_H) = (\hat{H}_0^{(\alpha)}, \hat{H}_0^{(\alpha)})_{\psi^{(\alpha)}_H}$, we see that $\mu_0(\psi^{(\alpha)}_H) = \Gamma(\alpha + 1/2)$ by (4.3.1.4). But, for $k = 0, 1, 2, \ldots$,

\[
\mu_{2k}(\psi^{(\alpha)}_H) = \int_{-\infty}^{\infty} x^{2k}|x|^{2\alpha}e^{-x^2} \, dx = \int_{-\infty}^{\infty} |x|^{2(\alpha+k)}e^{-x^2} \, dx = \mu_0(\psi^{(\alpha+k)}_H).
\]

Hence,

\[
\mu_{2k}(\psi^{(\alpha)}_H) = \Gamma\left(\alpha + k + \frac{1}{2}\right), \ k = 0, 1, 2, \ldots.
\] (4.3.1.6)

The explicit formulas (4.3.1.1) and (4.3.1.2) together are only one of many ways to describe the monic generalized Hermite polynomials. They
can be given by the generating function

\[
\frac{1 + 2xr + 4r^2}{(1 + 4r^2)^{\alpha + \frac{3}{2}}} \exp \left( \frac{4x^2r^2}{1 + 4r^2} \right) = \sum_{n=0}^{\infty} \frac{2^n}{\left[ \frac{n}{2} \right]!} \hat{H}_n^{(\alpha)}(x) r^n
\]  

(4.3.1.7)

(see [Chi], (2.49), p. 158), or the fundamental recurrence formula

\[
\hat{H}_n^{(\alpha)}(x) = x\hat{H}_{n-1}^{(\alpha)}(x) - \lambda_n \hat{H}_{n-2}^{(\alpha)}(x), \quad n = 1, 2, 3, \ldots,
\]  

(4.3.1.8)

where

\[
\lambda_n = \begin{cases} 
  k + \alpha - 1/2, & \text{if } n = 2k \\
  k, & \text{if } n = 2k + 1
\end{cases}
\]

(see [Chi], (2.46), p. 157). Also, there is a kind of Rodrigues’ type formula (see [Chi], (2.48), p.157) and a system of differential equations ([Chi], (2.44), p. 157) for these polynomials.

For each choice of \( \alpha > -1/2 \), the Transformation Theorem applied to the system of monic generalized Hermite polynomials of parameter \( \alpha \) results in a system of monic orthogonal Laurent polynomials, for each choice of parameters \( \lambda > 0 \) and \( \gamma > 0 \). The L-polynomials, which we denote by \( \tilde{H}_m^{(\alpha)}(x) \), have the explicit expressions

\[
\tilde{H}_{4k}^{(\alpha)}(x) = (-1)^k k! \; x^{2k} \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k - j} \frac{(-1)^j}{j! \lambda^{2j}} \; (x^2 - \gamma)^{2j} \; x^{2(k-j)},
\]  

\[ k = 0, 1, 2, \ldots, \]  

(4.3.1.9)
\[ \tilde{H}_{4k+1}(x) = (-1)^k k! \left( \frac{\lambda}{\gamma} \right)^{2k} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \binom{k + \alpha - 1/2}{k - j} \frac{(-1)^j}{j! \lambda^{2j}} (x^2 - \gamma)^{2j} x^{2(k-j)}, \]
\[ k = 0, 1, 2, \ldots, \quad (4.3.1.10) \]

\[ \tilde{H}_{4k+2}(x) = (-1)^k k! \lambda^{2k+1} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \binom{k + \alpha + 1/2}{k - j} \frac{(-1)^j}{j! \lambda^{2j+1}} (x^2 - \gamma)^{2j+1} x^{2(k-j)}, \]
\[ k = 0, 1, 2, \ldots, \quad (4.3.1.11) \]

and

\[ \tilde{H}_{4k+3}(x) = (-1)^{k+1} k! \left( \frac{\lambda}{\gamma} \right)^{2k+1} \frac{1}{x^{2(k+1)}} \sum_{j=0}^{k} \binom{k+\alpha+1/2}{k-j} \frac{(-1)^j}{j! \lambda^{2j+1}} (x^2 - \gamma)^{2j+1} x^{2(k-j)}, \]
\[ k = 0, 1, 2, \ldots, \quad (4.3.1.12) \]

given by equations (4.3.1.1) and (4.3.1.2) and Theorem 2.3.1 (C). According to the Transformation Theorem, Theorem 3.2.1 and equation (4.3.1.3), \( \{ \tilde{H}_m(x) \}_{m=0}^{\infty} \) is the monic OLPS with respect to a SMDF, which we denote by \( \tilde{\psi}_H^{(\alpha)} \), defined by

\[ \frac{d \tilde{\psi}_H^{(\alpha)}}{dx} = \frac{1}{\lambda^{2\alpha}} \left| x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right|^\alpha \exp \left( -\frac{1}{\lambda^2} \left( x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right) \right), \quad x \in \mathbb{R}^- \cup \mathbb{R}^+. \]
\[ (4.3.1.13) \]
By (4.3.1.13), (4.3.1.4) and Theorem 2.2.8, the orthogonality relation is

\[
\langle \tilde{H}_m^{(\alpha)}, \tilde{H}_n^{(\alpha)} \rangle_{\psi_H^{(\alpha)}} = \int_{-\infty}^{0} \tilde{H}_m^{(\alpha)}(x)\tilde{H}_n^{(\alpha)}(x) \frac{1}{\lambda^{2\alpha}} \left| x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right|^\alpha e^{-\frac{1}{\lambda^2}(x^2-2\gamma+\frac{\gamma^2}{x^2})} \, dx
\]
\[
+ \int_{0}^{\infty} \tilde{H}_m^{(\alpha)}(x)\tilde{H}_n^{(\alpha)}(x) \frac{1}{\lambda^{2\alpha}} \left| x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right|^\alpha e^{-\frac{1}{\lambda^2}(x^2-2\gamma+\frac{\gamma^2}{x^2})} \, dx
\]
\[= k_m \delta_{mn}, \quad (4.3.1.14)\]

where

\[
k_m = \begin{cases} 
\frac{\lambda^{2j+1}}{\gamma^{2j+1}} \left( \frac{j}{2} \right)! \Gamma \left( \frac{j+1}{2} + \alpha + \frac{1}{2} \right), & \text{if } m = 2j \\
\frac{\lambda^{2j+1}}{\gamma^{2j+1}} \left( \frac{j}{2} \right)! \Gamma \left( \frac{j+1}{2} + \alpha + \frac{1}{2} \right), & \text{if } m = 2j + 1.
\end{cases} \quad (4.3.1.15)
\]

Considering Theorem 2.2.5 (B), Theorem 3.3.5 and equations (4.3.1.5) and (4.3.1.6), we can find expressions for the moments of \( \tilde{\psi}_H^{(\alpha)} \):

\[
\mu_{2l+1}(\tilde{\psi}_H^{(\alpha)}) = 0, \quad l = 0, \pm 1, \pm 2, \ldots, \quad (4.3.1.16)
\]

\[
\mu_{-2l-2}(\tilde{\psi}_H^{(\alpha)}) = \gamma^{-2l-1} \mu_{2l}(\tilde{\psi}_H^{(\alpha)}), \quad l = 0, 1, 2, \ldots, \quad (4.3.1.17)
\]

and

\[
\mu_{2l}(\tilde{\psi}_H^{(\alpha)}) = \sum_{i=0}^{l} \binom{2l-i}{i} \gamma^i \lambda^{2l-2i+1} \Gamma(\alpha + l - i + 1/2), \quad l = 0, 1, 2, \ldots. \quad (4.3.1.18)
\]

Lastly in this section, we mention that there are many alternate descriptions besides the explicit expressions (4.3.1.1) and (4.3.1.2) of the monic gen-
eralized Hermite polynomials which may be transferred to the monic OLPS \( \{ \tilde{H}_m^{(\alpha)}(x) \}_{m=0}^{\infty} \). For example, we can employ Theorem 3.9.1 and the generating function (4.3.1.7) to write

\[
G \left( \frac{1}{\lambda} (x - \gamma x), \lambda r^2 \right) + \frac{r}{x} G \left( \frac{1}{\lambda} (x - \gamma x), -\frac{\lambda}{\gamma} r^2 \right) = \sum_{m=0}^{\infty} b_m \tilde{H}_m^{(\alpha)}(x) r^m,
\]

where

\[
G(x, r) = \frac{1 + 2xr + 4r^2}{(1 + 4r^2)^{\alpha + \frac{3}{2}}} \exp \left( \frac{4x^2r^2}{1 + 4r^2} \right)
\]

and

\[
b_{2n+1} = b_{2n} = \frac{2^n}{\left[ \frac{n}{2} \right]!}, \quad n = 0, 1, 2, \ldots
\]

We can also use Theorem 3.5.3 and the recurrence relation (4.3.1.8) to find the fundamental recurrence formula for \( \{ \tilde{H}_m^{(\alpha)}(x) \}_{m=0}^{\infty} \):

\[
\tilde{H}_m^{(\alpha)}(x) = \alpha_m \tilde{H}_{m-1}^{(\alpha)}(x) + x(-1)^m \tilde{H}_{m-2}^{(\alpha)}(x) + \gamma_m \tilde{H}_{m-3}^{(\alpha)}(x) + \delta_m \tilde{H}_{m-4}^{(\alpha)}(x),
\]

where we take \( \tilde{H}_k^{(\alpha)}(x) \equiv 0 \) for \( k < 0 \) and where \( \alpha_m, \gamma_m \) and \( \delta_m \) are given by

\[
\alpha_{2n} = (-\gamma)^n, \quad \alpha_{2n+1} = 0, \quad (4.3.1.23)
\]

\[
\gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\gamma)^n, \quad (4.3.1.24)
\]

\[
\delta_{2n} = -\lambda^2 \lambda_n, \quad \delta_{2n+1} = -\left(\frac{\lambda}{\gamma}\right)^2 \lambda_n, \quad (4.3.1.25)
\]

and

\[
\lambda_n = \begin{cases} 
k + \alpha - 1/2, & \text{if } n = 2k \\
k, & \text{if } n = 2k + 1
\end{cases}
\]

(4.3.1.26)
for \( n = 1, 2, 3, \ldots \)

### 4.3.2. The Hermite Polynomials

The monic Hermite polynomials, which we denote by \( \hat{H}_n(x) \), are the monic generalized Hermite polynomials of parameter \( \alpha = 0 \):

\[
\hat{H}_n(x) = \hat{H}_n^{(0)}(x), \quad n = 0, 1, 2, \ldots
\]  \hspace{1cm} (4.3.2.1)

\( \{\hat{H}_n(x)\}_{n=0}^{\infty} \) is the monic OPS with respect to the MDF \( \psi_H \) which is given by

\[
\frac{d\psi_H}{dx} = e^{-x^2}, \quad x \in \mathbb{R},
\]  \hspace{1cm} (4.3.2.2)

according to (4.3.2.1) and (4.3.1.3). The orthogonality relation is

\[
(\hat{H}_m, \hat{H}_n)_{\psi_H} = \int_{-\infty}^{\infty} \hat{H}_m(x)\hat{H}_n(x) e^{-x^2} \, dx
\]

\[
= \left[ \frac{n}{2} \right]! \Gamma \left( \left\lceil \frac{n+1}{2} \right\rceil + \frac{1}{2} \right) \delta_{mn}
\]

\[
= \frac{n!}{2^n \sqrt{\pi}} \delta_{mn},
\]  \hspace{1cm} (4.3.2.3)

considering (4.3.2.1) and (4.3.1.4). By (4.3.2.2), (4.3.1.5) and (4.3.1.6), we find that the moments of \( \psi_H \) are given by

\[
\mu_{2k+1}(\psi_H) = 0, \quad k = 0, 1, 2, \ldots
\]  \hspace{1cm} (4.3.2.4)

and

\[
\mu_{2k}(\psi_H) = \Gamma \left( k + \frac{1}{2} \right) = \frac{(2k)!}{4^k (k!)^2} \sqrt{\pi}, \quad k = 0, 1, 2, \ldots
\]  \hspace{1cm} (4.3.2.5)
The monic Hermite polynomials can be defined in several ways. There is a simple generating function which is sometimes used as the basic definition from which other characterizations are derived:

\[ e^{2x r - r^2} = \sum_{n=0}^{\infty} \frac{2^n}{n!} \hat{H}_n(x) \ r^n \] (4.3.2.6)

(see [Chi], eq. (2.14), p. 145, for example). These polynomials are also given by the Rodrigues’ type formula ([Chi], (2.13), p. 145)

\[ \hat{H}_n(x) = \frac{1}{(-2)^n e^{-x^2}} \frac{d^n}{dx^n} e^{-x^2}, \ n = 0, 1, 2, \ldots. \] (4.3.2.7)

They are the eigenfunctions for a self-adjoint differential operator:

\[ \frac{d}{dx} \left( e^{-x^2} \frac{d}{dx} \hat{H}_n(x) \right) + 2ne^{-x^2} H_n(x) = 0, \ -\infty < x < \infty, \ n = 0, 1, 2, \ldots. \] (4.3.2.8)

According to (4.3.2.1) and the recurrence relation (4.3.1.8), the fundamental recurrence formula for the monic Hermite polynomials is

\[ \hat{H}_n(x) = x\hat{H}_{n-1}(x) - \frac{n-1}{2} \hat{H}_{n-2}(x), \ n = 1, 2, 3, \ldots. \] (4.3.2.9)

By (4.3.2.1) and the explicit expressions (4.3.1.1) and (4.3.1.2),

\[ \hat{H}_{2k}(x) = (-1)^k k! \sum_{j=0}^{k} \binom{k - 1/2}{k - j} \frac{(-1)^j}{j!} x^{2j}, \ k = 0, 1, 2, \ldots, \] (4.3.2.10)

and
\[ \hat{H}_{2k+1}(x) = (-1)^k k! \sum_{j=0}^{k} \left( \frac{k + 1/2}{k - j} \right) \frac{(-1)^j}{j!} x^{2j+1}, \quad k = 0, 1, 2, \ldots \] (4.3.2.11)

The monic Hermite polynomials can also be given by the single explicit formula

\[ \hat{H}_n(x) = \frac{n!}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)! k!}, \quad n = 0, 1, 2, \ldots \] (4.3.2.12)

([Chi], eq. (2.15), p. 146). Hence,

\[ \hat{H}_n(x) = \sum_{j=0}^{n} \hat{H}_{n,j} x^j, \quad n = 0, 1, 2, \ldots \] (4.3.2.13)

where

\[ \hat{H}_{n,j} := \begin{cases} \frac{(-1/4)^{n-j} n!}{j! \left( \frac{n-j}{2} \right)!}, & \text{if } n-j \text{ is even} \\ 0, & \text{otherwise} \end{cases} \] (4.3.2.14)

We can use the Transformation Theorem and its consequences to transfer (4.3.2.1) through (4.3.2.14) to a system of orthogonal Laurent polynomials. We denote the L-polynomials by \( \tilde{H}_m(x) \), and, considering (4.3.2.1),

\[ \tilde{H}_m(x) = \tilde{H}_m^{(0)}(x), \quad m = 0, 1, 2, \ldots \] (4.3.2.15)

\( \{\tilde{H}_m(x)\}_{m=0}^{\infty} \) is the monic OLPS with respect to the SMDF \( \tilde{\psi}_H = \tilde{\psi}_H^{(0)} \), given by

\[ \frac{d \tilde{\psi}_H}{dx} = e^{-\frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{x^2}{2}}, \quad x \in \mathbb{R}^+ \cup \mathbb{R}^- \] (4.3.2.16)
according to (4.3.1.13) with $\alpha = 0$. The orthogonality relation is

$$\langle \tilde{H}_m, \tilde{H}_n \rangle_{\psi_H} = \int_{-\infty}^{0} \tilde{H}_m(x)\tilde{H}_n(x) e^{-\frac{1}{\lambda^2} \left(x^2 - 2\gamma + \frac{\gamma^2}{\lambda^2}\right)} dx + \int_{0}^{\infty} \tilde{H}_m(x)\tilde{H}_n(x) e^{-\frac{1}{\lambda^2} \left(x^2 - 2\gamma + \frac{\gamma^2}{\lambda^2}\right)} dx$$

$$= k_m \delta_{mn},$$

where

$$k_m = \begin{cases} \lambda^{2j+1} \frac{j!}{2^j} \sqrt{\pi}, & \text{if } m = 2j \\ \left(\frac{\lambda}{2}\right)^{2j+1} \frac{j!}{2^j} \sqrt{\pi}, & \text{if } m = 2j + 1 \end{cases} \quad (4.3.2.18)$$

by Theorem 2.2.8 and the orthogonality relation (4.3.2.3). (4.3.1.16) through (4.3.1.18) with $\alpha = 0$ yield explicit expressions for the moments of $\tilde{\psi}_H$:

$$\mu_{2l+1}(\tilde{\psi}_H) = 0, \ l = 0, \pm 1, \pm 2, \ldots, \quad (4.3.2.19)$$

$$\mu_{-2l-2}(\tilde{\psi}_H) = \gamma^{-2l-1} \mu_{2l}(\tilde{\psi}_H), \ l = 0, 1, 2, \ldots, \quad (4.3.2.20)$$

and

$$\mu_{2l}(\tilde{\psi}_H) = \sqrt{\pi} \sum_{i=0}^{l} \binom{2l-i}{i} \gamma^i \lambda^{2l-2i+1} \frac{(2l-2i)!}{4^{l-i} (l-i)!}, \ l = 0, 1, 2, \ldots. \quad (4.3.2.21)$$

Considering equations (4.3.2.6) through (4.3.2.14), there are many ways to describe the Laurent polynomials of the OLPS $\{\tilde{H}_m(x)\}_{m=0}^{\infty}$. Using the generating function (4.3.2.6) and Theorem 3.9.1, one arrives at
\[
\exp \left( 2 \left( x - \frac{\gamma}{x} \right) r^2 - \lambda^2 r^4 \right) + \frac{r}{x} \exp \left( 2 \left( \frac{1}{x} - \frac{x}{\gamma} \right) r^2 - \frac{\lambda^2}{\gamma^2} r^4 \right) = \sum_{m=0}^{\infty} b_m \tilde{H}_m(x) r^m, \\
\text{(4.3.2.22)}
\]

where

\[b_{2n+1} = b_{2n} = \frac{2^n}{n!}, \ n = 0, 1, 2, \ldots \text{ (4.3.2.23)}\]

With \(v(x) := \frac{1}{x} \left( x - \frac{2}{x} \right)\), the L-polynomials of even L-degree are given by

\[
\tilde{H}_{2n}(x) = \frac{\lambda^n}{(-2)^n e^{-v^2(x)}} \frac{d^n}{dv^n(x)} e^{-v^2(x)}, \ n = 0, 1, 2, \ldots, \\
\text{(4.3.2.24)}
\]

according to Theorem 3.7.1 and the Rodrigues’ type formula (4.3.2.7), and they satisfy the system of differential equations

\[
\frac{d}{dx} \left( e^{-v^2(x)} \frac{d}{dx} \tilde{H}_{2n}(x) \right) + 2n v'(x) e^{-v^2(x)} \tilde{H}_{2n}(x) = 0,
\]

\[0 < |x| < \infty, \ n = 0, 1, 2, \ldots, \\
\text{(4.3.2.25)}
\]

by Theorem 3.8.1 and the system (4.3.2.8). Among the recurrence relations given in Section 3.5 for \(\{\tilde{H}_m(x)\}_{m=0}^{\infty}\) is the fundamental recurrence formula

\[
\tilde{H}_m(x) = \alpha_m \tilde{H}_{m-1}(x) + x^{-1} \tilde{H}_{m-2}(x) + \gamma_m \tilde{H}_{m-3}(x) + \delta_m \tilde{H}_{m-4}(x),
\]

\[m = 2, 3, 4, \ldots, \\
\text{(4.3.2.26)}
\]

where \(\tilde{H}_k(x) \equiv 0\) for \(k < 0\) and where \(\alpha_m, \gamma_m\) and \(\delta_m\) satisfy, for \(n = 1, 2, 3, \ldots,\)

\[
\alpha_{2n} = (-\gamma)^n, \quad \alpha_{2n+1} = 0, \\
\text{(4.3.2.27)}
\]
\[ \gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\frac{1}{\gamma})^n, \quad (4.3.2.28) \]
\[ \delta_{2n} = \frac{1-n}{2} \lambda^2 \text{ and } \delta_{2n+1} = \frac{1-n}{2} (\frac{\lambda}{\gamma})^2, \quad (4.3.2.29) \]

which results by applying Theorem 3.5.3, considering the recurrence formula (4.3.2.9). By Theorem 2.3.1 (C) and the explicit formulas (4.3.2.10) and (4.3.2.11), or by equations (4.3.1.9) through (4.3.1.12) with \( \alpha = 0 \), we also have

\[ \tilde{H}_{4k}(x) = (-1)^k \ k! \ \lambda^{2k} \frac{1}{x^2k} \sum_{j=0}^{k} \left( k - 1/2 \right) \frac{(-1)^j}{j!} \lambda^{2j} (x^2 - \gamma)^{2j} x^{2(k-j)}, \quad k = 0, 1, 2, \ldots, \quad (4.3.2.30) \]
\[ \tilde{H}_{4k+1}(x) = (-1)^k \ k! \left( \frac{\lambda}{\gamma} \right)^{2k+1} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \left( k - 1/2 \right) \frac{(-1)^j}{j!} \lambda^{2j} (x^2 - \gamma)^{2j} x^{2(k-j)}, \quad k = 0, 1, 2, \ldots, \quad (4.3.2.31) \]
\[ \tilde{H}_{4k+2}(x) = (-1)^k \ k! \lambda^{2k+1} \frac{1}{x^{2k+1}} \sum_{j=0}^{k} \left( k + 1/2 \right) \frac{(-1)^j}{j!} \lambda^{2j+1} (x^2 - \gamma)^{2j+1} x^{2(k-j)}, \quad k = 0, 1, 2, \ldots, \quad (4.3.2.32) \]

and
\[ \tilde{H}_{4k+3}(x) = (-1)^{k+1} \ k! \left( \frac{\lambda}{\gamma} \right)^{2k+1} \frac{1}{x^{2(k+1)}} \sum_{j=0}^{k} \left( k + 1/2 \right) \frac{(-1)^j}{j!} \lambda^{2j+1} (x^2 - \gamma)^{2j+1} x^{2(k-j)}, \quad k = 0, 1, 2, \ldots \quad (4.3.2.33) \]

We can find alternate explicit expressions for these L-polynomials by considering the formulas (4.3.2.12) through (4.3.2.14). First, by Theorem 2.3.1
\( \tilde{H}_{2n}(x) = \lambda^n \frac{n!}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k \left( \frac{2}{\lambda} (x - \frac{\gamma}{x}) \right)^{n-2k}}{(n-2k)! k!}, \quad n = 0, 1, 2, \ldots \) \hspace{1cm} (4.3.2.34)

and

\( \tilde{H}_{2n+1}(x) = \left( -\frac{\lambda}{\gamma} \right)^n \frac{n!}{2^n} \frac{1}{x} \sum_{k=0}^{[n/2]} \frac{(-1)^k \left( \frac{2}{\lambda} (x - \frac{\gamma}{x}) \right)^{n-2k}}{(n-2k)! k!}, \quad n = 0, 1, 2, \ldots \) \hspace{1cm} (4.3.2.35)

Lastly, by Theorem 3.6.1 and equations (4.3.2.13) and (4.3.2.14),

\[ \tilde{H}_{2n}(x) = \sum_{k=-n}^{n} \tilde{H}_{2n,k} x^k, \quad n = 0, 1, 2, \ldots, \] \hspace{1cm} (4.3.2.36)

and

\[ \tilde{H}_{2n+1}(x) = \sum_{k=-n-1}^{n-1} (-\gamma)^{-n} \tilde{H}_{2n,k+1} x^k, \quad n = 0, 1, 2, \ldots, \] \hspace{1cm} (4.3.2.37)

where

\[ \tilde{H}_{2n,k} = \sum_{0 \leq i \leq j \leq n} \left( \frac{j}{i} \right) (b-a)^{n-j} (-ab)^{j-i} \tilde{H}_{n,j}, \] \hspace{1cm} (4.3.2.38)

and where \( \hat{H}_{n,j} \) is given by (4.3.2.14) as

\[ \hat{H}_{n,j} := \begin{cases} \frac{(-1/4)^{n-j} n!}{j! (n-j)!^n}, & \text{if } n - j \text{ is even} \\ 0, & \text{otherwise} \end{cases} \]
4.4. The Generalized Laguerre Class

The systems of orthogonal polynomials associated with the names of Jacobi, Hermite, and Laguerre are collectively called the classical orthogonal polynomials. In this section we finish our treatment, begun in Section 4.2, of applying the Tranformation Theorem and some of its consequences to the systems of classical polynomials by considering those polynomials associated with Laguerre.

The monic Sonine-Laguerre or generalized Laguerre polynomials of parameter $\alpha > -1$ are denoted by $\hat{L}_n^{(\alpha)}(x)$, $n = 0, 1, 2, \ldots$. These polynomials can be defined by the Rodrigues’ type formula

$$
\hat{L}_n^{(\alpha)}(x) = \frac{1}{(-1)^n x^\alpha e^{-x}} \frac{d^n}{dx^n} \left( x^{n+\alpha} e^{-x} \right), \quad n = 0, 1, 2, \ldots,
$$

([Chi], equations (2.10) and (2.12), page 145). They satisfy the system of differential equations

$$
\frac{d}{dx} \left( x^{\alpha+1} e^{-x} \frac{d}{dx} \hat{L}_n^{(\alpha)}(x) \right) + n x^{\alpha} e^{-x} \hat{L}_n^{(\alpha)}(x) = 0, \quad 0 < x < \infty, \quad n = 0, 1, 2, \ldots
$$

([Chi], (2.17), p. 146). \( \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \) is the monic OPS with respect to the MDF, which we denote by $\psi_L^{(\alpha)}$, given by

$$
\frac{d \psi_L^{(\alpha)}}{dx} = \begin{cases} 
  x^{\alpha} e^{-x}, & \text{if } x \in \mathbb{R}^+, \\
  0, & \text{otherwise}.
\end{cases}
$$

(4.4.3)
The orthogonality relation is
\[
(\mathcal{L}^{(\alpha)}_m, \mathcal{L}^{(\alpha)}_n)_{\psi^{(\alpha)}} = \int_0^{\infty} \mathcal{L}^{(\alpha)}_m(x) \mathcal{L}^{(\alpha)}_n(x) x^\alpha e^{-x} \, dx = n! \, \Gamma(n + \alpha + 1) \, \delta_{mn}
\]
(4.4.4)
([Chi], (2.18), p. 148, and (2.12), p. 145).

We can find an expression of the moments of \(\psi^{(\alpha)}_L\) by observing that integration by parts gives
\[
\mu_n(\psi^{(\alpha)}_L) = \int_0^{\infty} x^{n+\alpha} e^{-x} \, dx
\]
\[
= \int_0^{\infty} (n + \alpha) x^{n+\alpha-1} e^{-x} \, dx
\]
\[
= (n + \alpha) \, \mu_{n-1}(\psi^{(\alpha)}_L)
\]
\[
= (\alpha + 1) \, \mu_n(\psi^{(\alpha)}_L),
\]
where \((\alpha + 1)_n\) is Pochhammer’s symbol. By (4.4.4) with \(m = n = 0\), \(\mu_0(\psi^{(\alpha)}_L) = \Gamma(\alpha + 1)\). Hence,
\[
\mu_n(\psi^{(\alpha)}_L) = (\alpha + 1)_n \, \Gamma(\alpha + 1), \, n = 0, 1, 2, \ldots \quad (4.4.5)
\]

Many characterizations of the monic generalized Laguerre polynomials of parameter \(\alpha > -1\) are known. Perhaps most revealing are the connections to other sequences of orthogonal polynomials. As examples,
\[
\hat{C}^{(\alpha)}_n(x) = (-1)^n \hat{L}^{(x-n)}(\alpha), \, n = 0, 1, 2, \ldots \quad (4.4.6)
\]
([Chi], (1.5), p. 171, and (2.12), p. 145) where \(\hat{C}^{(\alpha)}_n(x)\) are the monic Poisson
polynomials described in Section 4.1, and

\[ \hat{H}_{2n}^{(\alpha)}(x) = \hat{L}_n^{(\alpha-1/2)}(x^2), \quad n = 0, 1, 2, \ldots, \quad (4.4.7) \]

and

\[ \hat{H}_{2n+1}^{(\alpha)}(x) = x \hat{L}_n^{(\alpha+1/2)}(x^2), \quad n = 0, 1, 2, \ldots, \quad (4.4.8) \]

([Chi], (2.43), p. 156, (2.16), p. 146, and (2.12), p. 145) where \( \hat{H}_n^{(\alpha)}(x) \) are the monic generalized Hermite polynomials discussed in Section 4.3. Other defining characterizations for the monic generalized Laguerre polynomials of parameter \( \alpha > -1 \) include a generating function:

\[ \frac{1}{(1-r)^{\alpha+1}} \exp \frac{-xr}{1-r} = \sum_{n=0}^{\infty} (-1)^n n! \hat{L}_n^{(\alpha)}(x) r^n \quad (4.4.9) \]

([Chi], (2.38), p. 155, and (2.12), p. 145). The fundamental recurrence formula is

\[ \hat{L}_n^{(\alpha)}(x) = (x-2n-\alpha+1)\hat{L}_{n-1}^{(\alpha)}(x) - (n-1)(n+\alpha-1)\hat{L}_{n-2}^{(\alpha)}(x), \quad n = 1, 2, 3, \ldots \quad (4.4.10) \]

([Chi], (2.31), p. 154). Lastly, and perhaps simplest of all, we mention that these polynomials are given by the explicit expressions

\[ \hat{L}_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n = 0, 1, 2, \ldots \quad (4.4.11) \]

([Chi], (2.11) and (2.12), p. 154).
As mentioned at the end of Chapter 2, applying the Transformation Theorem to an MDF results in an SMDF, for each choice of \( \lambda > 0 \) and \( \gamma > 0 \), with part of the spectrum on the negative reals and part on the positive reals. This raises serious doubts as to the appropriateness of the resulting SMDF as an analogue to the corresponding MDF whose spectrum is contained, say, in the non-negative reals. One may consider such MDF’s as examples of the limitations of this transformation. The Poisson class discussed in Section 4.1 gives examples in this direction. The generalized Laguerre class here provides a second collection of such MDF’s, since we see that the resulting SMDF, which we denote \( \tilde{\psi}^{(\alpha)}_L \), is given by

\[
\frac{d\tilde{\psi}^{(\alpha)}_L}{dx} = \begin{cases} 
\left( \frac{1}{\lambda} \left( x - \frac{\alpha}{\lambda} \right) \right)^{\alpha} e^{-\frac{1}{\lambda} \left( x - \frac{\alpha}{\lambda} \right)}, & \text{if } x \in (-\sqrt{\gamma}, 0) \cup (\sqrt{\gamma}, \infty) \\
0, & \text{if } x \in (-\infty, -\sqrt{\gamma}] \cup (0, \sqrt{\gamma}] \end{cases}
\] (4.4.12)

according to (4.4.3) and Theorem 3.2.1. It is evident from the definitions that the spectrum \( \sigma(\tilde{\psi}^{(\alpha)}_L) \) of \( \tilde{\psi}^{(\alpha)}_L \) is \( [-\sqrt{\gamma}, 0) \cup [\sqrt{\gamma}, \infty) \), while \( \sigma(\psi^{(\alpha)}_L) = [0, \infty) \).

We will return to the problem of Laguerre-like strong moment distributions in the next section of this chapter.

Regardless of our views about the question of appropriateness in this case, we can proceed to transfer the formulas for the monic generalized Laguerre polynomials of parameter \( \alpha > -1 \) over to the corresponding OLPS given by the Transformation Theorem. We denote the L-polynomials by
\( \tilde{L}^{(\alpha)}_m(x) \), and we find the explicit expressions

\[
\tilde{L}^{(\alpha)}_{2n}(x) = (-1)^n n! \lambda^n \frac{1}{x^n} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-1)^k}{k! \lambda^k} (x^2 - \gamma)^k x^{n-k}, \quad n = 0, 1, 2, \ldots,
\]

and

\[
\tilde{L}^{(\alpha)}_{2n+1}(x) = n! \left( \frac{\lambda}{\gamma} \right)^n \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-1)^k}{k! \lambda^k} (x^2 - \gamma)^k x^{n-k},
\]

\[
n = 0, 1, 2, \ldots,
\]

(4.4.13)

by (4.4.11) and Theorem 2.3.1 (C). Considering (4.4.12), the inner-products (4.4.4) and Theorem 2.2.8, the orthogonality relation is

\[
(\tilde{L}^{(\alpha)}_m, \tilde{L}^{(\alpha)}_n)_{\tilde{\psi}^{(\alpha)}} = \int_{-\sqrt{\gamma}}^{0} \tilde{L}^{(\alpha)}_m(x) \tilde{L}^{(\alpha)}_n(x) \left( \frac{1}{\lambda} (x - \frac{\gamma}{x}) \right)^{\alpha} e^{-\frac{1}{\lambda} (x - \frac{\gamma}{x})} dx
\]

\[
+ \int_{\sqrt{\gamma}}^{\infty} \tilde{L}^{(\alpha)}_m(x) \tilde{L}^{(\alpha)}_n(x) \left( \frac{1}{\lambda} (x - \frac{\gamma}{x}) \right)^{\alpha} e^{-\frac{1}{\lambda} (x - \frac{\gamma}{x})} dx
\]

\[
= k_m \delta_{mn},
\]

(4.4.15)

where

\[
k_m = \begin{cases} 
\lambda^{2j+1} (\alpha + 1)_j \Gamma(\alpha + 1), & \text{if } m = 2j \\
(\frac{\lambda}{\gamma})^{2j+1} (\alpha + 1)_j \Gamma(\alpha + 1), & \text{if } m = 2j + 1.
\end{cases}
\]

(4.4.16)

Expressions for the moments of \( \tilde{\psi}^{(\alpha)}_L \) can be found by combining (4.4.5), Theorem 2.2.5 and Theorem 3.3.5:

\[
\mu_{-1}(\tilde{\psi}^{(\alpha)}_L) = 0,
\]

(4.4.17)

\[
\mu_n(\tilde{\psi}^{(\alpha)}_L) = (-1)^n \gamma^{n+1} \mu_{n-2}(\tilde{\psi}^{(\alpha)}_L), \quad n = 0, \pm 1, \pm 2, \ldots,
\]

(4.4.18)
and

\[ \mu_n(\widetilde{\psi}_L^{(\alpha)}) = \Gamma(\alpha + 1) \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \gamma^k \lambda^{n-2k+1}(\alpha + 1)_{n-2k}, \quad n = 0, 1, 2, \ldots \]

(4.4.19)

From the Rodrigues’ type formula (4.4.1), the system of differential equations (4.4.2), the relations (4.4.6), (4.4.7) and (4.4.8), the generating function (4.4.9) and the fundamental recurrence formula (4.4.10) come alternate descriptions to those given by the explicit expressions (4.4.13) and (4.4.14) of the Laurent polynomials of the OLPS in this case. For example, the fundamental recurrence formula for \(\{\widetilde{L}_m^{(\alpha)}(x)\}_{m=0}^{\infty}\) is

\[
\widetilde{L}_m^{(\alpha)}(x) = \alpha_m \widetilde{L}_{m-1}^{(\alpha)}(x) + (x(-1)^m + \beta_m) \widetilde{L}_{m-2}^{(\alpha)}(x) + \gamma_m \widetilde{L}_{m-3}^{(\alpha)}(x) + \delta_m \widetilde{L}_{m-4}^{(\alpha)}(x), \quad m = 2, 3, 4, \ldots ,
\]

(4.4.20)

where we take \(\widetilde{L}_k^{(\alpha)}(x) \equiv 0\) for \(k < 0\) and where \(\alpha_m, \beta_m, \gamma_m\) and \(\delta_m\) are given by

\[
\alpha_{2n} = (-\gamma)^n, \quad \alpha_{2n+1} = 0, \quad (4.4.21)
\]

\[
\beta_{2n} = -\lambda (2n + \alpha - 1), \quad \beta_{2n+1} = \frac{\lambda}{\gamma} (2n + \alpha - 1), \quad (4.4.22)
\]

\[
\gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\frac{1}{\gamma})^n, \quad (4.4.23)
\]

\[
\delta_{2n} = -\lambda^2(n - 1)(n + \alpha - 1) \text{ and } \delta_{2n+1} = -(\frac{\lambda}{\gamma})^2(n - 1)(n + \alpha - 1), \quad (4.4.24)
\]

for \(n = 1, 2, 3, \ldots\), according to Theorem 3.5.3 and the recurrence relation (4.4.10).
4.5. Laguerre-Type Strong Distributions
Beyond the Range of the Transformation

Here, we further explore the uses and limitations of the Transformation
Theorem. In Section 4.5.1 we will exploit connections between the classical
Hermite and Laguerre systems in order to derive, as an indirect result of the
Transformation Theorem, a more suitable OLPS analogue for the classical
Laguerre system than that discussed in the last section. Then, in Section
4.5.2, we present some results on an SMDF which evidently is beyond even
the indirect reach of our transformation, yet seemingly is a natural Laurent
polynomial analogue of the classical Laguerre MDF.

4.5.1. A Laguerre Class Related to the Generalized Hermite Class. Let
$\alpha > -1/2$ and define $\{A_m^{(\alpha)}(x)\}_{m=0}^{\infty}$ by

$$A^{(\alpha)}_{2n}(x^2) := \tilde{H}^{(\alpha)}_{4n}(x), \ n = 0, 1, 2, \ldots, \quad (4.5.1.1)$$

and

$$A^{(\alpha)}_{2n+1}(x^2) := \tilde{H}^{(\alpha)}_{4n+3}(x), \ n = 0, 1, 2, \ldots \quad (4.5.1.2)$$

By (4.5.1.1) and (4.3.1.9),

$$A^{(\alpha)}_{2n}(x) = (-1)^n \ n! \ \lambda^{2n} \frac{1}{x^n} \sum_{j=0}^{n} \left( \frac{n + \alpha - 1/2}{n - j} \right) (-1)^j \frac{1}{j! \lambda^{2j}} (x - \gamma)^{2j} x^{n-j},$$

$$n = 0, 1, 2, \ldots, \quad (4.5.1.3)$$
and by (4.5.1.2) and (4.3.1.12),

\[ A^{(\alpha)}_{2n+1}(x) = (-1)^{n+1} n! \left( \frac{\lambda}{\gamma} \right)^{2n+1} \frac{1}{x^{n+1}} \sum_{j=0}^{n} \binom{n + \alpha + 1/2}{n - j} \frac{(-1)^j}{j! \lambda^{2j+1}} (x - \gamma)^{2j+1} x^{n-j}, \]

\( n = 0, 1, 2, \ldots. \) \hspace{1cm} (4.5.1.4)

Inspection of (4.5.1.3) and (4.5.1.4) shows that \( A^{(\alpha)}_m(x) \) is a monic L-polynomial of L-degree \( m \), for each \( m = 0, 1, 2, \ldots. \)

Next, define \( \phi_A^{(\alpha)}(x) \) by

\[ \frac{d\phi_A^{(\alpha)}}{dx} := \begin{cases} \frac{1}{\lambda^{\alpha}} \left| x - \frac{\gamma^2}{x} \right|^\alpha \exp \left( -\frac{1}{\lambda^{2\alpha}} \left( x - 2\gamma + \frac{\gamma^2}{x} \right) \right), & \text{if } x \in \mathbb{R}^+ \\ 0, & \text{if } x \in \mathbb{R}^- \end{cases} \]

Considering (4.5.1.5), \( \phi_A^{(\alpha)} \) is a SMDF with spectrum \( \sigma(\phi_A^{(\alpha)}) = \mathbb{R}^+ \), whose moments are

\[ \mu_l(\phi_A^{(\alpha)}) = \mu_{2l}(\tilde{\psi}_H^{(\alpha)}), \quad l = 0, \pm 1, \pm 2, \ldots, \] \hspace{1cm} (4.5.1.6)

by the change of variables \( x^2 \to x \) in

\[ \mu_{2l}(\tilde{\psi}_H^{(\alpha)}) = 2 \int_0^\infty x^{2l} \frac{1}{\lambda^{2\alpha}} \left| x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right|^\alpha e^{-\frac{1}{\lambda^{2\alpha}} \left( x^2 - 2\gamma + \frac{\gamma^2}{x^2} \right)} dx. \]

If \( k, l, m \) and \( n \) are non-negative integers such that \( A^{(\alpha)}_m(x^2) = \tilde{H}^{(\alpha)}(x) \) and \( A^{(\alpha)}_n(x^2) = \tilde{H}^{(\alpha)}(x) \), then, by linearity of the integral and (4.5.1.6),

\[ (A^{(\alpha)}_m, A^{(\alpha)}_n)_{\phi_A^{(\alpha)}} = (\tilde{H}^{(\alpha)}, \tilde{H}^{(\alpha)})_{\tilde{\psi}_H^{(\alpha)}}. \] \hspace{1cm} (4.5.1.7)

Hence, \( \{A^{(\alpha)}_m(x)\}_{m=0}^{\infty} \) is the monic OLPS with respect to \( \phi_A^{(\alpha)} \), the ortho-
nality relation being

\[(A^{(\alpha)}_m, A^{(\alpha)}_n) \psi_A^{(\alpha)} \]

\[= \int_0^\infty A^{(\alpha)}_m(x)A^{(\alpha)}_n(x) \frac{1}{\lambda^{2\alpha}} x^{-\frac{3}{2}} \left| x - 2\gamma + \frac{\gamma^2}{x} \right|^{\alpha} e^{-\frac{1}{\lambda^2} \left( x - 2\gamma + \frac{\gamma^2}{x} \right)} \, dx \]

\[= k_m \delta_{mn}, \quad (4.5.1.8)\]

where

\[k_m = \begin{cases} 
\lambda^{4j+1} (j!) \Gamma \left( \left[ \frac{2j+1}{2} \right] + \alpha + \frac{1}{2} \right), & \text{if } m = 2j \\
\left( \frac{\lambda}{\gamma} \right)^{4j+3} \left( \left[ \frac{2j+1}{2} \right] ! \right) \Gamma \left( j + \alpha + \frac{3}{2} \right), & \text{if } m = 2j + 1 \end{cases} \quad (4.5.1.9)\]

Inspection of the explicit expressions (4.5.1.3) and (4.5.1.4) reveals that
each of the L-polynomials of the monic OLPS \( \{A^{(\alpha)}_m(x)\}_{m=0}^\infty \) is regular.

Hence, in contrast to the other Laurent polynomial analogues of the classical orthogonal polynomials studied, \( \{A^{(\alpha)}_m(x)\}_{m=0}^\infty \) is given by a three term recurrence formula of the form

\[A^{(\alpha)}_m(x) = \left( \frac{x^{(-1)}^m}{\beta_{m-1}} + \beta_m \right) A^{(\alpha)}_{m-1}(x) - \delta_m A^{(\alpha)}_{m-2}(x), \quad m = 1, 2, 3, \ldots, \quad (4.5.1.10)\]

where we take \( A^{(\alpha)}_k(x) \equiv 0 \) for \( k < 0 \) and where \( \{\beta_n\}_{n=0}^\infty \) and \( \{\delta_m\}_{m=1}^\infty \) are sequences of real numbers (see [NT], equation (2.13), page 64). By observing that

\[A^{(\alpha)}_{2n}(x^2) = \lambda^{2n} \tilde{H}^{(\alpha)}_{2n}(v(x)), \quad n = 0, 1, 2, \ldots, \quad (4.5.1.11)\]
and

$$A_{2n+1}^{(\alpha)}(x^2) = \left(-\frac{\lambda}{\gamma}\right)^{2n+1} \frac{1}{x} \tilde{H}_{2n+1}^{(\alpha)}(v(x)), \quad n = 0, 1, 2, \ldots,$$

(4.5.12)

which follow from (4.5.1.1) and (4.5.1.2), and by recalling the recurrence formula (4.3.1.8) for \(\{\tilde{H}_n^{(\alpha)}(x)\}_{n=0}^\infty\), we can find that the coefficients in the relation (4.5.1.10) satisfy

$$\beta_n = (-\gamma)^{(-1)^n n}, \quad n = 0, 1, 2, \ldots,$$

(4.5.13)

and

$$\delta_m = \begin{cases} \lambda^2 (k + \alpha - 1/2), & \text{if } m = 2k \\ (\lambda/\gamma)^2 k, & \text{if } m = 2k + 1 \end{cases}, \quad m = 1, 2, 3, \ldots$$

(4.5.14)

4.5.2. A Natural Laguerre-Type Strong Distribution. By taking \(\alpha = 0\) and \(\lambda = 1\) in the case studied in the previous section, we may obtain a system of orthogonal Laurent polynomials whose SMDF, \(\phi_A(x) := e^{-2\gamma \phi_A^{(0)}(x)}\), is given by

$$\frac{d\phi_A}{dx} = \begin{cases} x^{-1/2} e^{-x-(\gamma^2/x)}, & \text{if } x \in \mathbb{R}^+ \\ 0, & \text{if } x \in \mathbb{R}^- \end{cases},$$

(4.5.2.1)

for each choice of \(\gamma > 0\). Thus, the class of OLPS systems studied in the previous section intersects, at one point, the class of systems with SMDF \(\phi_L^{(\alpha,\beta)}\), where we define \(\phi_L^{(\alpha,\beta)}\) by

$$\frac{d\phi_L^{(\alpha,\beta)}}{dx} := \begin{cases} x^{\alpha} e^{-x-(\beta/x)}, & \text{if } x \in \mathbb{R}^+ \\ 0, & \text{if } x \in \mathbb{R}^- \end{cases}, \quad \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{R}^+. \quad (4.5.2.2)$$
It is easy to check that \( \phi_L^{(\alpha,\beta)} \) is an SMDF. For example, the integrals defining the moments of \( \phi_L^{(\alpha,\beta)} \) are easily seen to be bounded by the moments of the SMDF given in (4.5.2.1). Recalling the classical generalized Laguerre distribution \( \psi_L^{(\alpha)} \) given by (4.4.3), we consider \( \phi_L^{(\alpha,\beta)} \) given by (4.5.2.2) to be a natural SMDF analogue of \( \psi_L^{(\alpha)} \): The spectra and weight functions seem to be in greater harmony with one another, especially in comparison to the previously discussed cases. However, \( \phi_L^{(\alpha,\beta)} \) appears to be well beyond the range of our work based on the doubling transformation \( \nu(x) = \frac{1}{x}(x - \frac{2}{x}) \). Despite this, we next make an effort to complete the work of finding Laurent polynomial analogues for the three classes of classical orthogonal polynomials by finding expressions for the moments of \( \phi_L^{(\alpha,\beta)} \).

The moments of an SMDF have a variety of uses in the general theory of systems of orthogonal Laurent polynomials. As an example, consider that if \( \{R_n(x)\}_{n=0}^{\infty} \) is the monic OLPS with respect to a SMDF \( \phi \) having moments \( \mu_n, \ n = 0, \pm 1, \pm 2, \ldots \), then

\[
R_{2m}(x) = \frac{1}{H_{2m}^{(-2m)}} \left| \begin{array}{cccc}
\mu_{-2m} & \mu_{-2m+1} & \cdots & \mu_{-1} \\
\mu_{-2m+1} & \mu_{-2m+2} & \cdots & \mu_0 \\
\vdots & \vdots & \ddots & \vdots \\
\mu_0 & \mu_1 & \cdots & \mu_{2m-1} \\
\end{array} \right| x^{-m}
\]

and

\[
R_{2m+1}(x) = \frac{1}{H_{2m+1}^{(-2m)}} \left| \begin{array}{cccc}
\mu_{-(2m+1)} & \mu_{-2m} & \cdots & \mu_{-1} \\
\mu_{-2m} & \mu_{-2m+1} & \cdots & \mu_0 \\
\vdots & \vdots & \ddots & \vdots \\
\mu_0 & \mu_1 & \cdots & \mu_{2m} \\
\end{array} \right| x^{-m}
\]
where, for \( n = 0, \pm 1, \pm 2, \ldots \), \( \mathcal{H}_0^{(n)} := 1 \) and

\[
\mathcal{H}_k^{(n)} := \begin{vmatrix}
\mu_n & \mu_{n+1} & \cdots & \mu_{n+k-1} \\
\mu_{n+1} & \mu_{n+2} & \cdots & \mu_{n+k} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n+k-1} & \mu_{n+k} & \cdots & \mu_{n+2k-2}
\end{vmatrix}, \quad k = 1, 2, 3, \ldots
\]

(see [CC], Theorem 2.5, page 57).

We can find expressions for the moments of \( \phi_L^{(\alpha, \beta)} \) in terms of the Laplace transform. The (one-dimensional) Laplace transform of a function \( F \) of a real variable \( t \) is a function, which we denote by \( \mathcal{L}\{F(t)\} \), of a complex variable \( s \), given by

\[
\mathcal{L}\{F(t)\}(s) := \int_0^\infty F(t) \, e^{-st} \, dt,
\]

whenever the integral converges. Thus, it follows from (4.5.2.2) that, for any Laurent polynomials \( R \) and \( S \),

\[
(R, S)_{\phi_L^{(\alpha, \beta)}} = \int_0^\infty R(t) \, S(t) \, t^\alpha e^{-\beta/t} \, e^{-t} \, dt = \mathcal{L}\{R(t) \, S(t) \, t^\alpha e^{-\beta/t}\}\{1\}.
\]

(4.5.2.5)

For example, if we let \( \{L_m^{(\alpha, \beta)}(x)\}_{m=0} \) denote the monic OLPS with respect to \( \phi_L^{(\alpha, \beta)} \), then

\[
(L_m^{(\alpha, \beta)}, L_m^{(\alpha, \beta)})_{\phi_L^{(\alpha, \beta)}} = \mathcal{L}\{\left(L_m^{(\alpha, \beta)}(t)\right)^2 \, t^\alpha e^{-\beta/t}\}\{1\}, \quad m = 0, 1, 2, \ldots.
\]

(4.5.2.6)
Also, we find

\[ \mu_l(\phi_{L}^{(\alpha, \beta)}) = \int_0^\infty t^{1+\alpha} e^{-\beta/t} e^{-t} \, dt = \mathcal{L}\{t^{1+\alpha} e^{-\beta/t}\}(1), \ l = 0, \pm 1, \pm 2, \ldots. \]

(4.5.2.7)

In particular, for \( \phi_{L}^{(0, \beta)} \), given by

\[
\frac{d\phi_{L}^{(0, \beta)}}{dx} := \begin{cases} 
    e^{-x-(\beta/x)}, & \text{if } x \in \mathbb{R}^+, \\
    0, & \text{if } x \in \mathbb{R}^-,
\end{cases} \beta \in \mathbb{R}^+,
\]

we have

\[ \mu_l(\phi_{L}^{(0, \beta)}) = \mathcal{L}\{t^{l} e^{-\beta/t}\}(1) = 2 \ (\sqrt{\beta})^{l+1} K_{l+1}(2\sqrt{\beta}), \ l = 0, 1, 2, \ldots. \]

(4.5.2.8)

where \( K_\nu \) is the function known as the modified Bessel function of order \( \nu \) (See [AS], Section 9.6.1, p. 374). The substitution \( t \rightarrow \frac{2}{t} \) shows that

\[ \mu_l(\phi_{L}^{(0, \beta)}) = \beta^{l+1} \mu_{-l-2}(\phi_{L}^{(0, \beta)}), \ l = 0, 1, 2, \ldots, \]

(4.5.2.9)

from which it follows that

\[ \mu_l(\phi_{L}^{(0, \beta)}) = 2 \ (\sqrt{\beta})^{l+1} K_{l+1}(2\sqrt{\beta}), \ l = -2, -3, -4, \ldots. \]

(4.5.2.10)

It is also evident that

\[ \mu_{-1}(\phi_{L}^{(0, \beta)}) = 2 K_0(2\sqrt{\beta}). \]

(4.5.2.11)
Thus, (4.5.2.8) holds for all integers $l$; that is,

$$
\mu_l (\phi_L^{(0, \beta)}) = \mathcal{L}\{ t^l e^{-\beta/t} \}(1) = 2 (\sqrt{\beta})^{l+1} K_{l+1}(2\sqrt{\beta}), \quad l = 0, \pm 1, \pm 2, \ldots,
$$

(4.5.2.12)