3.1. Discrete Distribution Functions

A function \( f : D \to \mathbb{R} \) is called a \textit{jump function} on \( D \subseteq \mathbb{R} \) if there exists a sequence \( \{x_n\}_{n=1}^{\infty} \subseteq D \) and a sequence \( \{j_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+_0 \) such that \( \sum_{n=1}^{\infty} j_n < \infty \) and \( f(x) = \sum_{n=1}^{\infty} j_n I_{[x_n, \infty)}(x) \) for all \( x \in D \), where

\[
I_A(x) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}
\]

is the indicator function for a set \( A \). In this case we say \( f \) is a jump function on \( D \) with jumps \( j_n \) at \( x_n \). A MDF \( \psi \) is called \textit{discrete} if \( \psi \) is a jump function on \( \mathbb{R} \). A SMDF \( \phi \) is called \textit{discrete} if \( \phi|_{\mathbb{R}^+} \) is a jump function on \( \mathbb{R}^+ \) and if \( \phi|_{\mathbb{R}^-} \) is a jump function on \( \mathbb{R}^- \).

\textbf{Theorem 3.1.1.} Suppose \( \psi \) is a discrete MDF given by jumps \( j_n \) at \( x_n \). Then \( \tilde{\psi} \) is a discrete SMDF given by jumps \( \frac{j_n}{v'(v_{\pm}^{-1}(x_n))} \) at \( v_{\pm}^{-1}(x_n) \).
Proof: Suppose $x < 0$. Then, by the substitution $t = v^{-1}(y)$,

$$\tilde{\psi}(x) : = \int_{-\infty}^{x} \frac{1}{v'(t)} d(\psi \circ v)(t)$$

$$= \int_{-\infty}^{v(x)} \frac{1}{v'(v^{-1}(y))} d\psi(y)$$

$$= \int_{-\infty}^{\infty} \frac{I(-\infty,v(x))(y)}{v'(v^{-1}(y))} d\psi(y)$$

$$= \sum_{n=1}^{\infty} \frac{I_{(-\infty,v(x))}(x_n)}{v'(v^{-1}(x_n))} j_n.$$  

Since $x < 0$, we have $x_n \leq v(x)$ if and only if $v^{-1}(x_n) \leq x$. Hence,

$I_{(-\infty,v(x))}(x_n) = I_{[v^{-1}(x_n),\infty)}(x)$, and

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \frac{j_n}{v'(v^{-1}(x_n))} I_{[v^{-1}(x_n),\infty)}(x) \text{ for } x < 0.$$  

From this it is easy to verify that $\tilde{\psi}|_{\mathbb{R}^-}$ is a jump function on $\mathbb{R}^-$ with jumps $\frac{j_n}{v'(v^{-1}(x_n))}$ at $v^{-1}(x_n)$. In a similar manner it can be shown that $\tilde{\psi}|_{\mathbb{R}^+}$ is a jump function on $\mathbb{R}^+$ with jumps $\frac{j_n}{v'(v^+(x_n))}$ at $v^+(x_n)$. \(\square\)

3.2. Weight Functions

We restrict our attention in this section to differentiable distribution functions. If $\psi$ is an MDF which is differentiable, then $w(x) = \frac{d\psi}{dx}$ is called the weight function for $\psi$. Similarly, if $\phi$ is an SMDF which is differentiable, then $\omega(x) = \frac{d\phi}{dx}$ is called the weight function for $\phi$. 
Theorem 3.2.1. If $\psi$ is an MDF, and if $\psi$ is differentiable with $w(x) := \frac{d\psi}{dx}$, then $\tilde{\psi}$ is differentiable with $\frac{d\tilde{\psi}}{dx} = w(v(x))$.

Proof: If $x < 0$,

$$\tilde{\psi}(x) = \int_{-\infty}^{\infty} \frac{1}{v'(t)} d(\psi \circ v)(t)$$

$$= \int_{-\infty}^{\infty} \frac{1}{v'(t)} w(v(t))v'(t) dt$$

$$= \int_{-\infty}^{\infty} w(v(t)) dt,$$

and, by the fundamental theorem of calculus, $\frac{d\tilde{\psi}}{dx}$ exists and equals $w(v(x))$.

Similarly, if $x > 0$, $\frac{d\tilde{\psi}}{dx}$ exists and equals $w(v(x))$. $\Box$

3.3. Moments

Our purpose here is to forge explicit formulas for the moments with respect to $\tilde{\psi}$ in terms of the moments with respect to $\psi$, and vice versa. The key to our sequence of results is the following theorem which is a relation between inner-products.

Theorem 3.3.1. $\mu_n(\psi) = \frac{1}{n} (v^n, 1)_\psi$, for $n = 0, 1, 2, \ldots$.

Proof: Let $n$ be a non-negative integer. By Theorem 2.2.7,

$$(v^n, 1)_\psi = \int_{-\infty}^{0} v^n(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \int_{0}^{\infty} v^n(x) \frac{1}{v'(x)} d(\psi \circ v)(x).$$
The substitution \( x \to -\frac{\gamma}{t} \) gives

\[
(v^n, 1)_\psi = \int_0^{\infty} v^n(-\frac{\gamma}{t}) \frac{1}{v'(x)|_{x=-\frac{\gamma}{t}}} d(\psi \circ v)(-\frac{\gamma}{t}) + \\
\int_{-\infty}^{0} v^n(-\frac{\gamma}{t}) \frac{1}{v'(x)|_{x=-\frac{\gamma}{t}}} d(\psi \circ v)(-\frac{\gamma}{t}).
\]

Theorem 2.2.1, parts (D) and (E), then imply

\[
(v^n, 1)_\psi = \int_0^{\infty} v^n(t) \frac{\gamma}{t^2} \frac{1}{v'(x)|_{x=t}}\ d(\psi \circ v)(t) + \\
\int_{-\infty}^{0} v^n(t) \frac{\gamma}{t^2} \frac{1}{v'(x)|_{x=t}}\ d(\psi \circ v)(t).
\]

Thus, we see that

\[
(v^n, 1)_\psi = \int_{-\infty}^{0} v^n(x) \frac{\gamma}{x^2} \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \int_0^{\infty} v^n(x) \frac{\gamma}{x^2} \frac{1}{v'(x)}\ d(\psi \circ v)(x).
\]

Hence,

\[
2 (v^n, 1)_\psi = \int_{-\infty}^{0} v^n(x) \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \int_0^{\infty} v^n(x) \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \\
\int_{-\infty}^{0} v^n(x) \frac{\gamma}{x^2} \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \int_0^{\infty} v^n(x) \frac{\gamma}{x^2} \frac{1}{v'(x)}\ d(\psi \circ v)(x)
\]

\[
= \lambda \int_{-\infty}^{0} v^n(x) \frac{1}{\lambda} \left(1 + \frac{\gamma}{x^2}\right) \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \\
\lambda \int_0^{\infty} v^n(x) \frac{1}{\lambda} \left(1 + \frac{\gamma}{x^2}\right) \frac{1}{v'(x)}\ d(\psi \circ v)(x)
\]

\[
= \lambda \int_{-\infty}^{0} v^n(x)v'(x) \frac{1}{v'(x)}\ d(\psi \circ v)(x) + \\
\lambda \int_0^{\infty} v^n(x)v'(x) \frac{1}{v'(x)}\ d(\psi \circ v)(x)
\]

\[
= \lambda \int_{-\infty}^{0} v^n(x) d(\psi \circ v)(x) + \lambda \int_0^{\infty} v^n(x) d(\psi \circ v)(x);
\]
that is,

\[(v^n, 1) \tilde{\psi} = \frac{\lambda}{2} \int_{-\infty}^{0} v^n(x) d(\psi \circ v)(x) + \frac{\lambda}{2} \int_{0}^{\infty} v^n(x) d(\psi \circ v)(x).\]

The substitution \(v(x) \rightarrow y\) now gives

\[(v^n, 1) \tilde{\psi} = \frac{\lambda}{2} \int_{-\infty}^{\infty} y^n d\psi(y) + \frac{\lambda}{2} \int_{-\infty}^{\infty} y^n d\psi(y);\]

that is, \((v^n, 1) \tilde{\psi} = \lambda \int_{-\infty}^{\infty} y^n d\psi(y) = \lambda \mu_n(\psi). \mu_n(\psi) = \frac{1}{\lambda} (v^n, 1) \tilde{\psi}\) therefore follows. □

**Theorem 3.3.2.** \(\mu_n(\psi) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi}),\) for \(n = 0, 1, 2, \ldots\)

**Proof:** Let \(n\) be a non-negative integer. By Theorem 3.3.1,

\[\mu_n(\psi) = \frac{1}{\lambda} (v^n, 1) \tilde{\psi}.\]

By the Binomial Theorem,

\[v^n(x) = \left(\frac{1}{\lambda} (x - \frac{\gamma}{x})\right)^n\]

\[= \frac{1}{\lambda^n} \sum_{k=0}^{n} \binom{n}{k} x^k (-\gamma)^{n-k}\]

\[= \frac{1}{\lambda^n} \sum_{k=0}^{n} \binom{n}{k} (-\gamma)^{n-k} x^{2k-n}.\]
Thus, by linearity of the integral,

$$
\mu(\psi) = \frac{1}{\lambda} (v^n, 1)_{\tilde{\psi}} = \frac{1}{\lambda} \frac{1}{\lambda^n} \sum_{k=0}^{n} \binom{n}{k} (-\gamma)^{n-k} (a^{2k-n}, 1)_{\tilde{\psi}};
$$

that is, $\mu_n(\psi) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi})$. □

The previous theorem gives the moments $\mu_n(\psi)$ in terms of the moments $\mu_{-n}(\tilde{\psi}), \mu_{-n+2}(\tilde{\psi}), \mu_{-n+4}(\tilde{\psi}), \ldots, \mu_n(\tilde{\psi})$. We use Theorem 2.2.5 next to find a formula for $\mu_n(\psi)$ in terms of non-negative moments with respect to $\tilde{\psi}$.

**Theorem 3.3.3.** $\mu_n(\psi) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{\left[\frac{n}{2}\right]-1} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi})$ for $n = 0, 1, 2, \ldots$.

**Proof:** Let $n$ be a non-negative integer. By Theorem 3.3.2,

$$
\mu_n(\psi) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi}).
$$

We therefore have

$$
\mu_n(\psi) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{n-\left[\frac{n}{2}\right]-1} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi}) + \frac{1}{\lambda^{n+1}} \sum_{k=n-\left[\frac{n}{2}\right]}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi}).
$$

We proceed by first noticing that, by reversing the order of summation
in the second summation in this last expression for \( \mu_n(\psi) \), we have

\[
\sum_{k=n-\left\lceil \frac{n}{2} \right\rceil}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi}) = \sum_{k=0}^{\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k} (-\gamma)^k \mu_{n-2k}(\tilde{\psi}).
\]

Next, we manipulate the first summation in the last expression for \( \mu_n(\psi) \).

By Theorem 2.2.5,

\[
\mu_{2k-n}(\tilde{\psi}) = (-1)^{2k-n} \gamma^{2k-n+1} \mu_{-(2k-n)-2}(\tilde{\psi}).
\]

Hence,

\[
\mu_{2k-n}(\tilde{\psi}) = (-1)^n \gamma^{2k-n+1} \mu_{n-2(k+1)}(\tilde{\psi}),
\]

and

\[
\sum_{k=0}^{n-\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\tilde{\psi})
\]

\[
= \sum_{k=0}^{n-\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k} (-\gamma)^{n-k} \left( (-1)^n \gamma^{2k-n+1} \mu_{n-2(k+1)}(\tilde{\psi}) \right)
\]

\[
= \sum_{k=0}^{n-\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k} (-1)^{k+1} \mu_{n-2(k+1)}(\tilde{\psi})
\]

\[
= - \sum_{k=1}^{n-\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k-1} (-\gamma)^k \mu_{n-2k}(\tilde{\psi}).
\]

Now, if \( n \) is an even non-negative integer, then \( n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor \), and if \( n \) is an odd non-negative integer, then \( \mu_{n-2(n-\left\lceil \frac{n}{2} \right\rceil)}(\tilde{\psi}) = \mu_{-1}(\tilde{\psi}) \), which is zero by Theorem 2.2.5(C). Hence, the upper limit of summation in the sum

\[
\sum_{k=1}^{n-\left\lceil \frac{n}{2} \right\rceil} \binom{n}{k-1} (-\gamma)^k \mu_{n-2k}(\tilde{\psi})
\]

can be replaced with \( \left\lceil \frac{n}{2} \right\rceil \). Also, since we have
taken \((\binom{n}{1})\) to be zero, the lower limit of summation can be replaced with 0.

We therefore have

\[
\sum_{k=0}^{n-\left[\frac{n}{2}\right]-1} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\widetilde{\psi}) = -\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k-1} (-\gamma)^{k}\mu_{n-2k}(\widetilde{\psi}).
\]

We finally return to the expression for \(\mu_n(\psi)\) to see that

\[
\mu_n(\psi) = \frac{1}{\chi^{n+1}} \sum_{k=0}^{n-\left[\frac{n}{2}\right]-1} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\widetilde{\psi}) + \frac{1}{\chi^{n+1}} \sum_{k=n-\left[\frac{n}{2}\right]}^{n} \binom{n}{k} (-\gamma)^{n-k} \mu_{2k-n}(\widetilde{\psi})
\]

\[
= \frac{1}{\chi^{n+1}} \left(-\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k-1} (-\gamma)^{k}\mu_{n-2k}(\widetilde{\psi}) + \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} (-\gamma)^{k}\mu_{n-2k}(\widetilde{\psi})\right)
\]

\[
= \frac{1}{\chi^{n+1}} \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(\binom{n}{k} - \binom{n}{k-1}\right) (-\gamma)^{k}\mu_{n-2k}(\widetilde{\psi}). \quad \Box
\]

We now endeavor to invert the formula of Theorem 3.3.3 in order to find an expression for the moments of \(\widetilde{\psi}\) in terms of those of \(\psi\). Our proof of our result will rely on the following identity for binomial coefficients. This identity may be contained in the literature. A proof is included here for completeness.

**Lemma 3.3.4.** \((\binom{s+i}{i}) = \sum_{k=1}^{i} \left(\binom{s+k}{k} - \binom{s+i}{k-1}\right) (-1)^{k+1}, for all natural numbers \(s\) and \(i\) satisfying \(1 \leq i \leq s\).
Proof: We proceed by induction on $s$. First, notice that

\[
\binom{s}{1} = s = \left( \binom{s+1}{1} - \binom{s+1}{0} \right) \left( \binom{s-1}{0} \right) (-1)^2,
\]

by which we infer that $\binom{s}{i} = \sum_{k=1}^{i} \left( \binom{s+i}{k} - \binom{s+i}{k-1} \right) \binom{s-k}{i-k} (-1)^{k+1}$ for $1 = i \leq s$, and, hence, the lemma is true for $s = 1$.

Assume $s \geq 2$ and

\[
\binom{m}{i} = \sum_{k=1}^{i} \left( \binom{m+i}{k} - \binom{m+i}{k-1} \right) \binom{m-k}{i-k} (-1)^{k+1}, \quad 1 \leq i \leq m < s.
\]

Notice that

\[
\sum_{k=1}^{s} \left( \binom{s+s}{k} - \binom{s+s}{k-1} \right) \binom{s-k}{s-k} (-1)^{k+1} = \sum_{k=1}^{s} \left( \binom{2s}{k} - \binom{2s}{k-1} \right) (-1)^{k+1}
\]

\[
= \sum_{k=1}^{s} \binom{2s}{k} (-1)^{k+1} - \sum_{k=1}^{s} \binom{2s}{k-1} (-1)^{k+1}
\]

\[
= \sum_{k=1}^{s} \binom{2s}{k} (-1)^{k+1} + \sum_{k=0}^{s-1} \binom{2s}{k} (-1)^{k+1}.
\]

By symmetry of the binomial coefficients,

\[
\sum_{k=0}^{s-1} \binom{2s}{k} (-1)^{k+1} = \sum_{k=s+1}^{2s} \binom{2s}{k} (-1)^{k+1}.
\]
Hence,

\[ \sum_{k=1}^{s} \left( \binom{s+s}{k} - \binom{s+s}{k-1} \right) \binom{s-k}{s-k} (-1)^{k+1} \]

\[ = \sum_{k=1}^{s} \binom{2s}{k} (-1)^{k+1} + \sum_{k=0}^{s-1} \binom{2s}{k} (-1)^{k+1}. \]

\[ = \sum_{k=1}^{s} \binom{2s}{k} (-1)^{k+1} + \sum_{k=s+1}^{2s} \binom{2s}{k} (-1)^{k+1} \]

\[ = \sum_{k=1}^{2s} \binom{2s}{k} (-1)^{k+1}. \]

Using the Binomial Theorem, one can easily show that \( \sum_{k=1}^{2s} \binom{2s}{k} (-1)^{k+1} = 1 - \sum_{k=0}^{2s} \binom{2s}{k} (-1)^{k} \) and \( \sum_{k=0}^{2s} \binom{2s}{k} (-1)^{k} = (1 - 1)^{2s} = 0. \) Thus,

\[ \sum_{k=1}^{s} \left( \binom{s+s}{k} - \binom{s+s}{k-1} \right) \binom{s-k}{s-k} (-1)^{k+1} = 1 = \binom{s}{s}; \]

that is, the lemma is true for \( i = s \). We’ve seen that the lemma is true for \( i = 1 \), hence we assume that

\[ 1 < i < s. \] (3.3.4.2)

To continue, we will employ the fact that \( \binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1} \), whenever \( 0 \leq j < n \). Then, with \( 1 < i < s \),

\[ \sum_{k=1}^{i} \left( \binom{s+i}{k} - \binom{s+i}{k-1} \right) \binom{s-k}{i-k} (-1)^{k+1} \]

\[ = \sum_{k=1}^{i} (-1)^{k+1} \left( \binom{s+i}{k} - \binom{s+i}{k-1} \right) \left( \binom{s-k-1}{i-k} + \binom{s-k-1}{i-k-1} \right) \]
\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( \begin{array}{c} s-k-1 \\ i-k \end{array} \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( s-k-1 \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i-1 \\ k-1 \end{array} \right) - \left( \begin{array}{c} s+i-1 \\ k-2 \end{array} \right) \right) \left( s-k-1 \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( s-k-1 \right). \]

By (3.3.4.1) with \( m = s-1 \), \( \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i-1 \\ k \end{array} \right) - \left( \begin{array}{c} s+i-1 \\ k-1 \end{array} \right) \right) \left( s-k-1 \right) = \left( \begin{array}{c} s-1 \\ i \end{array} \right). \) Thus,

\[ \sum_{k=1}^{i} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( s-k \right) \left( i-k \right) \left( s-1 \right) \]

\[ \left( \begin{array}{c} s-1 \\ i \end{array} \right) + \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i-1 \\ k-1 \end{array} \right) - \left( \begin{array}{c} s+i-1 \\ k-2 \end{array} \right) \right) \left( s-k-1 \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( s-k-1 \right). \]

Now,

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i \\ k \end{array} \right) - \left( \begin{array}{c} s+i \\ k-1 \end{array} \right) \right) \left( s-k-1 \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i-1 \\ k \end{array} \right) - \left( \begin{array}{c} s+i-1 \\ k-1 \end{array} \right) \right) \left( s-k-1 \right) \]

\[ \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \begin{array}{c} s+i-1 \\ k-1 \end{array} \right) - \left( \begin{array}{c} s+i-1 \\ k-2 \end{array} \right) \right) \left( s-k-1 \right) \]
Thus,

\[
\sum_{k=1}^{i} \left( \binom{s+i}{k} - \binom{s+i}{k-1} \right) \binom{s-k}{i-k} (-1)^{k+1}
\]

\[
= \binom{s-1}{i} + \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \binom{s+i-1}{k-1} - \binom{s+i-1}{k-2} \right) \binom{s-k-1}{i-k} \right)
\]

\[
= \binom{s-1}{i} + \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \binom{s+i-1}{k-1} - \binom{s+i-1}{k-2} \right) \binom{s-k-1}{i-k} \right)
\]

\[
= \binom{s-1}{i} + \sum_{k=1}^{i} (-1)^{k+1} \left( \left( \binom{s+i-1}{k-1} - \binom{s+i-1}{k-2} \right) \binom{s-k-1}{i-k} \right)
\]

\[
= \binom{s-1}{i} + \binom{s-1}{i-1} + \]

\[
\sum_{k=2}^{i} (-1)^{k+1} \left( \left( \binom{s+i-1}{k-1} - \binom{s+i-1}{k-2} \right) \binom{s-k}{i-k} \right) + \]

\[
\sum_{k=1}^{i} (-1)^{k+1} \left( \left( \binom{s+i-1}{k} - \binom{s+i-1}{k-1} \right) \binom{s-k-1}{i-k} \right)
\]
\[
\binom{s-1}{i} + \binom{s-1}{i-1} - \sum_{k=1}^{i} (-1)^{k+1} \left( \binom{s+i-1}{k} - \binom{s+i-1}{k-1} \right) \binom{s-k-1}{i-k-1}
\]

\[
+ \sum_{k=1}^{i} (-1)^{k+1} \left( \binom{s+i-1}{k} - \binom{s+i-1}{k-1} \right) \binom{s-k-1}{i-k-1}
\]

\[
= \binom{s-1}{i} + \binom{s-1}{i-1}
\]

\[
= \binom{s}{i},
\]

and the lemma is proved. \(\Box\)

We can now proceed in our formulation of the connection between moments by giving an expression for \(\mu_n(\tilde{\psi})\) in terms of the moments with respect to \(\psi\), when \(n\) is a non-negative integer. In view of Theorem 2.2.5, the next result suffices in giving all the moments of \(\tilde{\psi}\) in terms of the moments of \(\psi\).

**Theorem 3.3.5.** \(\mu_n(\tilde{\psi}) = \sum [\frac{n}{k} \gamma^k \lambda^{n-2k+1} \mu_{n-2k}(\psi)]\) for \(n = 0, 1, 2, \ldots\).

**Proof:** First, using Theorem 3.3.3,

\[
\mu_n(\tilde{\psi}) = \frac{1}{\lambda^{n+1}} \sum_{k=0}^{[\frac{n}{2}]} \left( \binom{n}{k} - \binom{n}{k-1} \right) \left( -\gamma \right)^k \mu_{n-2k}(\tilde{\psi})
\]

\[
= \frac{1}{\lambda^{n+1}} \left( \mu_n(\tilde{\psi}) + \sum_{k=1}^{[\frac{n}{2}]} \left( \binom{n}{k} - \binom{n}{k-1} \right) \left( -\gamma \right)^k \mu_{n-2k}(\tilde{\psi}) \right),
\]

hence

\[
\mu_n(\tilde{\psi}) = \lambda^{n+1} \mu_n(\psi) - \sum_{k=1}^{[\frac{n}{2}]} \left( \binom{n}{k} - \binom{n}{k-1} \right) \left( -\gamma \right)^k \mu_{n-2k}(\tilde{\psi}), \quad (3.3.5.1)
\]
for \( n = 0, 1, 2, \ldots \).

We proceed by induction on \( n \). Notice \( \mu_0(\tilde{\psi}) = \lambda \mu_0(\psi) \) by (3.3.5.1), and

\[
\lambda \mu_0(\psi) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \gamma^k \lambda^{n-2k+1} \mu_{n-2k}(\psi)
\]

when \( n = 0 \). Hence, the theorem is true for \( n = 0 \).

Next, suppose \( n \geq 1 \), and assume

\[
\mu_m(\tilde{\psi}) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \gamma^j \lambda^{m-2j+1} \mu_m(\psi), \quad 0 \leq m \leq n-1. \tag{3.3.5.2}
\]

Use of (3.3.5.2) in (3.3.5.1) yields

\[
\mu_n(\tilde{\psi}) = \lambda^{n+1} \mu_n(\psi) - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n}{k} - \binom{n}{k-1} \right) (-\gamma)^k \sum_{j=0}^{\lfloor \frac{n-2k-j}{2} \rfloor} \binom{n-2k-j}{j} \gamma^j \lambda^{n-2k-2j+1} \mu_{n-2k-2j}(\psi).
\]

Re-indexing over \( i = j + k \), we find that

\[
\mu_n(\tilde{\psi}) = \lambda^{n+1} \mu_n(\psi) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} A_{n,i} \gamma^i \lambda^{n-2i+1} \mu_{n-2i}(\psi)
\]

where

\[
A_{n,i} := \sum_{1 \leq k \leq i, \ k+j=i} \left( \binom{n}{k} - \binom{n}{k-1} \right) \binom{n-2k-j}{j} (-1)^{k+1}.
\]

Thus,

\[
A_{n,i} = \sum_{k=1}^{i} \left( \binom{n}{k} - \binom{n}{k-1} \right) \binom{n-i-k}{i-k} (-1)^{k+1},
\]

and, by Lemma 3.3.4, \( \binom{n-i}{i} = \sum_{k=1}^{i} \left( \binom{n}{k} - \binom{n}{k-1} \right) \binom{n-i-k}{i-k} (-1)^{k+1} \); that is,

\[
A_{n,i} = \binom{n-i}{i}.
\]
Hence,

\[ \mu_n(\tilde{\psi}) = \lambda^{n+1} \mu_n(\psi) + \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \gamma^i \lambda^{n-2i+1} \mu_{n-2i}(\psi) \]

\[ = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \gamma^i \lambda^{n-2i+1} \mu_{n-2i}(\psi). \square \]

It is worth noting that Theorem 3.3.5 implies, in the limiting case \( \lambda = 1 \) and \( \gamma = 0 \), that \( \mu_n(\tilde{\psi}) = \mu_n(\psi) \) for non-negative integers \( n \).

### 3.4. Zeros

A rich theory of the zeros of the polynomials in an OPS has been developed and used in the overall theory of OPS’s. To accomplish our modest goals in this section, we will implicitly use the fact that the zeros of \( P_n(x) \) are real and simple (for example, see [Chi], Theorem 5.2, page 27). In particular, we know that if \( n \) is a positive integer, then \( P_n(x) \) has \( n \) zeros, say \( x_{n1}, x_{n2}, \ldots, x_{nn} \), which can be ordered such that \( x_{n1} < x_{n2} < \cdots < x_{nn} \). We are therefore assured that the hypotheses of our next theorem are met.

**Theorem 3.4.1.** Let \( n \) be a positive integer, and suppose \( x_{n1}, x_{n2}, \ldots, x_{nn} \) are the zeros of \( P_n \) such that \( x_{n1} < x_{n2} < \cdots < x_{nn} \). Then the zeros of \( \tilde{P}_{2n}(x) \) and \( \tilde{P}_{2n+1}(x) \) are \( x_{nj}^\pm := v_\pm^{-1}(x_{nj}) \), for \( 1 \leq j \leq n \), and have the
ordering $x_{n1}^- < x_{n2}^- < \cdots < x_{nn}^- < 0 < x_{n1}^+ < x_{n2}^+ < \cdots < x_{nn}^+$. 

Proof: Let $z$ be any one of the real numbers $x_{nj}^\pm$, $1 \leq j \leq n$. By Theorem 2.2.1, part (B) or part (C), $v(z)$ is a zero of $P_n(x)$. Hence, $\tilde{P}_{2n}(z) := \lambda^n P_n(v(z)) = 0$, and, thus, each of the real numbers $x_{nj}^\pm$ is a zero of $\tilde{P}_{2n}(x)$. Similarly, each $x_{nj}^\pm$ is also a zero of $\tilde{P}_{2n+1}(x)$. The ordering $x_{n1}^- < x_{n2}^- < \cdots < x_{nn}^-$ and the monotonicity of $v_- : \mathbb{R} \to \mathbb{R}^+$ and $v_- : \mathbb{R} \to \mathbb{R}^-$ given by Theorem 2.2.1 (A) imply that $x_{n1}^- < x_{n2}^- < \cdots < x_{nn}^-$ and the monotonicity of $v_+ : \mathbb{R} \to \mathbb{R}^+$ and $v_+ : \mathbb{R} \to \mathbb{R}^+$ given by Theorem 2.2.1 (A) imply that $x_{n1}^- < x_{n2}^- < \cdots < x_{nn}^- < 0 < x_{n1}^+ < x_{n2}^+ < \cdots < x_{nn}^+$. Inspection of the definitions shows that $\tilde{P}_{2n}(x)$ and $\tilde{P}_{2n+1}(x)$ can have at most $2n$ zeros, thus the real numbers $x_{nj}^\pm$, for $1 \leq j \leq n$, are all of the zeros of $\tilde{P}_{2n}(x)$ and $\tilde{P}_{2n+1}(x)$. \(\square\)

3.5. Recurrence Formulas

It is known that orthogonal polynomial sequences satisfy a 3-term recurrence formula (for example, see [Chi], Theorem 4.1, page 18). In particular, if we define $P_{-1}(x) \equiv 0$, there exists real numbers $c_n$ and positive real numbers $\lambda_n$ such that

$$P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \ldots.$$ 

This relation is sometimes called the fundamental recurrence formula for the monic OPS \(\{P_n(x)\}_{n=0}^\infty\). Every monic OPS satisfies a unique relation of this
form, up to a choice of $\lambda_1$, and a rotund theory featuring connections between the coefficients $c_n$ and $\lambda_n$ and many aspects of the general theory of OPS’s has been developed. In many specific cases satisfactory explicit expressions for the coefficient $c_n$ and $\lambda_n$ are known. Our primary purpose in this section is to transfer, via the Transformation Theorem, the fundamental recurrence formula for the monic OPS $\{P_n(x)\}_{n=0}^\infty$ to the monic OLPS $\{\tilde{P}_m(x)\}_{m=0}^\infty$.

In general, if $\{R_m(x)\}_{m=0}^\infty$ is a monic OLPS, there are real numbers $\alpha_m$, $\beta_m$, $\gamma_m$ and $\delta_m$ such that $\{R_m(x)\}_{m=0}^\infty$ satifies the 5-term recurrence formula

$$R_m(x) = \alpha_m R_{m-1}(x) + (x^{-1})^m \beta_m R_{m-2}(x) + \gamma_m R_{m-3}(x) + \delta_m R_{m-4}(x),$$

where we set $R_k(x) \equiv 0$ for $k < 0$. See [CC], Theorem 4.5, page 71, for a proof. This relation is unique up to a choice of $\gamma_2$, $\delta_2$ and $\delta_3$, and it is sometimes called the fundamental recurrence formula for the monic OLPS $\{R_m(x)\}_{m=0}^\infty$. It is true that $\{R_m(x)\}_{m=0}^\infty$ is regular if and only if it satisfies a 3-term recurrence relation of a certain form (see [CC], Theorem 4.2, page 66, and Theorem 5.1, page 72). However, the following theorem implies that $\tilde{P}_{2n+1}(x)$ is singular, hence $\{\tilde{P}_m(x)\}_{m=0}^\infty$ is not regular.

**Theorem 3.5.1.** $\tilde{P}_{2n+1}(x) = (-\frac{1}{\gamma})^n \frac{1}{x} \tilde{P}_{2n}(x)$ for each non-negative integer $n$.

Proof: Evidently, $\tilde{P}_{2n+1}(x) := (-\frac{1}{\gamma})^n \frac{1}{x} P_n (v(x)) = (-\frac{1}{\gamma})^n \frac{1}{x} \lambda^{-n} \tilde{P}_{2n}(x) = (-\frac{1}{\gamma})^n \frac{1}{x} \tilde{P}_{2n}(x)$. □
Theorem 3.5.1 above and part (C) of the following theorem show that \( \{ \tilde{P}_m(x) \}_{m=0}^{\infty} \) is given by a 4-term recurrence relation.

**Theorem 3.5.2.** Suppose there exists constants \( c_n \in \mathbb{R} \) and \( \lambda_n \in \mathbb{R}^+ \) such that \( \{ P_n(x) \}_{n=0}^{\infty} \) satisfies the fundamental recurrence formula

\[
P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \ldots,
\]

where we define \( P_{-1}(x) \equiv 0 \). Let \( \tilde{P}_0(x) \equiv 1 \) and \( \tilde{P}_1(x) = \frac{1}{x} \). Then, for each positive integer \( n \):

(A) \( \tilde{P}_{2n}(x) = (x - \lambda c_n - \frac{\gamma}{x})\tilde{P}_{2n-2}(x) - \lambda^2 \lambda_n \tilde{P}_{2n-4}(x) \).

(B) \( \tilde{P}_{2n+1}(x) = \left( \frac{1}{x} + \frac{\lambda}{\gamma} c_n - \frac{1}{\gamma} \right)\tilde{P}_{2n-1}(x) - \left( \frac{\lambda}{\gamma} \right)^2 \lambda_n \tilde{P}_{2n-2}(x) \).

(C) \( \tilde{P}_{2n}(x) = (x - \lambda c_n - \frac{\gamma}{x})\tilde{P}_{2n-2}(x) - (\gamma)^{n-2} \lambda^2 \lambda_n x \tilde{P}_{2n-3}(x) \).

Proof: (A) The substitution \( x \to v(x) \) in the fundamental recurrence formula for \( \{ P_n(x) \}_{n=0}^{\infty} \), followed by the identification \( P_k(v(x)) = \lambda^{-k} \tilde{P}_{2k}(x) \) for \( k = n, n-1 \) and \( n-2 \), yields the result after multiplication by \( \lambda^n \).

(B) Similar to the proof of the part (A), the substitution \( x \to v(x) \) in the fundamental recurrence formula, followed by the identification \( P_k(v(x)) = (-\frac{\gamma}{x})^k x \tilde{P}_{2k}(x) \), yields the result after division by \( (-\frac{\gamma}{x})^n x \).

(C) An application of Theorem 3.5.1 to the left side of the equation in part (A) of Theorem 3.5.2 gives the stated result.

Lastly in this section, we report the 5-term fundamental recurrence formula for the monic OLPS \( \{ \tilde{P}_m(x) \}_{m=0}^{\infty} \).
Theorem 3.5.3. Suppose there exists constants $c_n \in \mathbb{R}$ and $\lambda_n \in \mathbb{R}^+$ such that \{\(P_n(x)\)\}$_{n=0}^{\infty}$ satisfies the fundamental recurrence formula
\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \ldots, \]
where we define $P_{-1}(x) \equiv 0$. Then \{\(\tilde{P}_m(x)\)\}$_{m=0}^{\infty}$ satisfies the fundamental recurrence formula
\[ \tilde{P}_m(x) = \alpha_m \tilde{P}_{m-1}(x) + (x^{-1}m + \beta_m)\tilde{P}_{m-2}(x) \]
\[ + \gamma_m \tilde{P}_{m-3}(x) + \delta_m \tilde{P}_{m-4}(x), \quad m = 2, 3, 4, \ldots, \]
where we define \(\tilde{P}_k(x) \equiv 0\) for $k < 0$ and where the coefficients $\alpha_m$, $\beta_m$, $\gamma_m$ and $\delta_m$ are given by
\[
\alpha_{2n} = (-\gamma)^n, \quad \alpha_{2n+1} = 0,
\]
\[
\beta_{2n} = -\lambda c_n, \quad \beta_{2n+1} = \frac{\lambda}{\gamma} c_n,
\]
\[
\gamma_{2n} = 0, \quad \gamma_{2n+1} = (-\frac{1}{\gamma})^n,
\]
\[
\delta_{2n} = -\lambda^2 \lambda_n \text{ and } \delta_{2n+1} = -\left(\frac{\lambda}{\gamma}\right)^2 \lambda_n,
\]
for $n = 1, 2, 3, \ldots$.

Proof: Using Theorem 3.5.2 (A), algebra and Theorem 3.5.1, we see that
\[
\tilde{P}_{2n}(x) = (x - \lambda c_n - \frac{\gamma}{x}) \tilde{P}_{2n-2}(x) - \lambda^2 \lambda_n \tilde{P}_{2n-4}(x)
\]
\[
= -\frac{\gamma}{x} \tilde{P}_{2n-2}(x) + (x - \lambda c_n) \tilde{P}_{2n-2}(x) - \lambda^2 \lambda_n \tilde{P}_{2n-4}(x)
\]
\[
= -\frac{\gamma}{x} \left( (-\gamma)^{n-1} x \tilde{P}_{2n-1}(x) \right) + (x - \lambda c_n) \tilde{P}_{2n-2}(x) - \lambda^2 \lambda_n \tilde{P}_{2n-4}(x)
\]
\[= (-\gamma)^n \tilde{P}_{2n-1}(x) + (x - \lambda c_n) \tilde{P}_{2n-2}(x) - \lambda^2 \lambda_n \tilde{P}_{2n-4}(x);\]
that is, $\alpha_{2n} = (-\gamma)^n$, $\beta_{2n} = -\lambda c_n$, $\gamma_{2n} = 0$ and $\delta_{2n} = -\lambda^2 \lambda_n$. In a similar
manner, we see that

\[ P_{2n+1}(x) \]

\[ = \left( \frac{1}{x} + \frac{\lambda}{\gamma} c_n - \frac{1}{\gamma} x \right) P_{2n-1}(x) - \left( \frac{\lambda}{\gamma} \right)^2 \lambda_n P_{2n-3}(x) \]

\[ = \left( \frac{1}{x} + \frac{\lambda}{\gamma} c_n \right) P_{2n-1}(x) - \frac{1}{\gamma} x P_{2n-1}(x) - \left( \frac{\lambda}{\gamma} \right)^2 \lambda_n P_{2n-3}(x) \]

\[ = \left( \frac{1}{x} + \frac{\lambda}{\gamma} c_n \right) P_{2n-1}(x) - \frac{1}{\gamma} x \left( \left( -\frac{1}{\gamma} \right)^{n-1} \frac{1}{x} \tilde{P}_{2n-2}(x) \right) - \left( \frac{\lambda}{\gamma} \right)^2 \lambda_n \tilde{P}_{2n-3}(x) \]

\[ = \left( \frac{1}{x} + \frac{\lambda}{\gamma} c_n \right) \tilde{P}_{2n-1}(x) + \left( -\frac{1}{\gamma} \right)^n \tilde{P}_{2n-2}(x) - \left( \frac{\lambda}{\gamma} \right)^2 \lambda_n \tilde{P}_{2n-3}(x); \]

that is, \( \alpha_{2n+1} = 0 \), \( \beta_{2n+1} = \frac{\lambda}{\gamma} c_n \), \( \gamma_{2n+1} = \left( -\frac{1}{\gamma} \right)^n \) and \( \delta_{2n+1} = -\left( \frac{\lambda}{\gamma} \right)^2 \lambda_n \). \( \square \)

### 3.6. L-polynomial Coefficients

**Theorem 3.6.1.** Let \( n \) be a non-negative integer, and suppose \( P_n(x) = \sum_{j=0}^{n} P_{n,j} x^j \) for real numbers \( P_{n,j} \), \( 0 \leq j \leq n \). Then:

(A) \( \tilde{P}_{2n}(x) = \sum_{k=-n}^{n} \tilde{P}_{2n,k} x^k \), where

\[ \tilde{P}_{2n,k} = \sum_{0 \leq i \leq j \leq n} \binom{j}{i} \lambda^{n-j} (-\gamma)^{j-i} P_{n,j}. \]

(B) \( \tilde{P}_{2n+1}(x) = \sum_{k=-n-1}^{n-1} \tilde{P}_{2n+1,k} x^k \), where \( \tilde{P}_{2n+1,k} = \left( -\frac{1}{\gamma} \right)^n \tilde{P}_{2n,k+1} \).
Proof: (A)

\[ \tilde{P}_{2n} = \lambda^n P(v(x)) \]

\[ = \lambda^n \sum_{j=0}^{n} P_{n,j} v^j(x) \]

\[ = \lambda^n \sum_{j=0}^{n} P_{n,j} \left( \frac{1}{\lambda} (x - \frac{\gamma}{x}) \right)^j \]

\[ = \lambda^n \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{j}{i} \lambda^i \left(\frac{-\gamma}{x}\right)^{j-i} P_{n,j} \]

\[ = \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{j}{i} \lambda^{n-j} \left(-\gamma\right)^{j-i} P_{n,j} x^{2i-j} \]

\[ = \sum_{k=-n}^{n} \tilde{P}_{2n,k} x^k, \]

where

\[ \tilde{P}_{2n,k} = \sum_{0 \leq i \leq j \leq n}^{2i-j = k} \binom{j}{i} \lambda^{n-j} \left(-\gamma\right)^{j-i} P_{n,j}. \]

(B) Employing Theorem 3.5.1 and Theorem 3.6.1 (A),

\[ \tilde{P}_{2n+1}(x) = (\frac{-1}{\gamma})^n \frac{1}{x} \tilde{P}_{2n}(x) \]

\[ = (\frac{-1}{\gamma})^n \frac{1}{x} \sum_{k=-n}^{n} \tilde{P}_{2n,k} x^k \]

\[ = \sum_{k=-n}^{n} (\frac{-1}{\gamma})^n \tilde{P}_{2n,k} x^{k-1} \]

\[ = \sum_{k=-n-1}^{n-1} (\frac{-1}{\gamma})^n \tilde{P}_{2n,k+1} x^k. \square \]
Theorem 3.6.2. For all non-negative integers \( n \), \( \tilde{P}_{2n}(x) \) is regular, and \( \tilde{P}_{2n+1}(x) \) is singular.

Proof: By Theorem 3.6.1 (A), the trailing coefficient of \( \tilde{P}_{2n}(x) \) is

\[
\tilde{P}_{2n,-n} = \binom{n}{0} \lambda^{n-n} (-\gamma)^{n-0} P_n,n = (-\gamma)^n,
\]
which isn’t 0. By Theorem 3.6.1 (B), the trailing coefficient of \( \tilde{P}_{2n+1}(x) \) is 0. □

3.7. Rodrigues’ Type Formulas

In several special cases of MDF’s given by weight functions, the monic OPS \( \{P_n(x)\}_{n=0}^{\infty} \) is given by a formula of the type

\[
P_n(x) = \frac{1}{K_n} \frac{d^n}{dx^n} (\rho^n(x) w(x)) , \quad n = 0, 1, 2, \ldots,
\]

where \( K_n \) is independent of \( x \), \( \rho(x) \) is a polynomial independent of \( n \) and \( w(x) \) is a (weight) function independent of \( n \) (for example, see [Chi], equation (2.17), page 146). A formula of this type is called a Rodrigues’ type formula.

Also, in several discrete MDF cases, there is a finite difference analogue of Rodrigues’ formula giving the monic OPS:

\[
P_n(x) = \frac{1}{K_n} \frac{1}{w(x)} (\Delta^n Y_n)(x) , \quad n = 0, 1, 2, \ldots,
\]
where $X(x)$ is a polynomial independent of $n$, $w(x)$ is independent of $n$ and $Y_n(x) := X(x)X(x-1) \cdots X(x-n+1)w(x-n)$ (see [Chi], equation (3.1), page 160, for example). Here, $\Delta^n$, given by

$$
(\Delta^n f)(x) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x + n - k)
$$

is called the $n$-th order finite difference operator. The following two theorems, the proofs of which are trivial, report the results of transferring these Rodrigues’ formulas via the Transformation Theorem.

**Theorem 3.7.1.** Let $n$ be any non-negative integer. Suppose

$$
P_n(x) = \frac{1}{K_n} \frac{d^n}{dx^n} (\rho^n(x) w(x)) .
$$

Then

$$
\tilde{P}_{2n}(x) = \frac{\lambda^n}{K_n} \frac{d^n}{d\nu^n(x)} (\rho^n(\nu(x)) \nu(w(x))) ,
$$

and

$$
\tilde{P}_{2n+1}(x) = \frac{(-\lambda)^n}{\gamma^n K_n x \nu(x)} \frac{d^n}{d\nu^n(x)} (\rho^n(\nu(x)) \nu(w(x))) .
$$

Proof: The definition $\tilde{P}_{2n}(x) := \lambda^n P_n(\nu(x))$ and the fact given by Theorem 3.5.1 that $\tilde{P}_{2n+1}(x) = \left(-\frac{1}{\gamma}\right)^n \frac{1}{x} \tilde{P}_{2n}(x)$ directly yield the stated results. □
Theorem 3.7.2. Let $n$ be any non-negative integer. Suppose

$$P_n(x) = \frac{1}{K_n} \frac{w(x)}{w(x)} \left(\Delta^n Y_n(x)\right).$$

Then

$$\tilde{P}_{2n}(x) = \frac{\lambda^n}{K_n} \frac{w(v(x))}{v(x)} \left(\Delta^n Y_n(v(x))\right),$$

and

$$\tilde{P}_{2n+1}(x) = \frac{(-\lambda)^n}{\gamma^n K_n} \frac{w(v(x))}{v(x)} \left(\Delta^n Y_n(v(x))\right).$$

Proof: The proof is the same as that for Theorem 3.7.1. □

3.8. Differential and Difference Equations

Closely related to Rodrigues’ formulas are second order differential and difference equations. For example, there are monic OPS’s which are the eigenfunctions for certain self-adjoint differential operators (see [Chi], equation (2.19), page 148). Since the L-polynomials of even L-degree given by the Transformation Theorem are obtained from the polynomials of a monic OPS by a change of variables, it is perhaps not suprising that these L-polynomials can be the eigenfunctions for a differential operator, as shown by the following theorem.
Theorem 3.8.1. Suppose $y = P_n(x)$ satisfies

$$\frac{d}{dx} \left( k(x) \frac{dy}{dx} \right) - \lambda_n w(x)y = 0, \ n = 0, 1, 2, \ldots,$$

for $x$ in the interior of $\sigma(\psi)$. Then $Y = \tilde{P}_{2n}(x)$ satisfies

$$\frac{d}{dx} \left( \frac{k(v(x))}{v'(x)} \frac{dY}{dx} \right) - \lambda_n w(v(x))v'(x)Y = 0, \ n = 0, 1, 2, \ldots,$$

for $x$ in the interior of $\sigma(\tilde{\psi})$.

Proof: Let $n$ be a non-negative integer. Then, for $x \in \sigma(\tilde{\psi})$ and $t = v(x)$, we have $t \in \sigma(\psi)$, and

$$\frac{d}{dx} \left( \frac{k(v(x))}{v'(x)} \frac{d}{dx} \tilde{P}_{2n}(x) \right) - \lambda_n w(v(x))v'(x)\tilde{P}_{2n}(x)$$

$$= \frac{d}{dx} \left( \frac{k(v(x))}{v'(x)} \frac{d}{dx} \lambda^n P_n(v(x)) \right) - \lambda_n w(v(x))v'(x)\lambda^n P_n(v(x))$$

$$= \frac{d}{dv(x)} \left( \frac{k(v(x))}{v'(x)} \left( \frac{d}{dv(x)} \lambda^n P_n(v(x)) \right) v'(x) \right) v'(x) - \lambda_n w(v(x))v'(x)\lambda^n P_n(v(x))$$

$$= \lambda^n v'(x) \left( \frac{d}{dv(x)} \left( k(v(x)) \frac{d}{dv(x)} P_n(v(x)) \right) - \lambda_n w(v(x))P_n(v(x)) \right)$$

$$= \lambda^n v'(x) \left( \frac{d}{dt} \left( k(t) \frac{d}{dt} P_n(t) \right) - \lambda_n w(t)P_n(t) \right)$$

$$= \lambda^n v'(x) (0)$$

$$= 0. \Box$$

A result for second order difference equations could be derived in a similar manner.
3.9. Generating Functions

A generating function for a sequence \( \{f_n(x)\}_{n=0}^{\infty} \) of functions is a function \( G(x, r) \) of two variables that has a formal power series expansion of the form

\[
G(x, r) = \sum_{n=0}^{\infty} a_n f_n(x) \ r^n,
\]

for a known sequence \( \{a_n\}_{n=0}^{\infty} \) of constants. Generating functions have long been used to define systems of OPS’s and to derive their properties (for example, see [Jac]). The next theorem shows that generating functions for OPS’s can be used to define generating functions for the corresponding OLPS’s given by the Transformation Theorem.

**Theorem 3.9.1.** Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of constants, and suppose \( G \) is a function of two variables such that

\[
G(x, r) = \sum_{n=0}^{\infty} a_n P_n(x) \ r^n
\]

as a formal power series in \( r \). Let \( H(x, r) \) and \( \{b_m\}_{m=0}^{\infty} \) be defined by

\[
H(x, r) := G\left(v(x), \lambda r^2\right) + \frac{r}{x} G\left(v(x), -\frac{\lambda}{\gamma} r^2\right)
\]

and

\[
b_{2n+1} := b_{2n} := a_n, \ n = 0, 1, 2, \ldots.
\]
Then

\[ H(x, r) = \sum_{m=0}^{\infty} b_m \tilde{P}_m(x) \, r^m \]

as a formal power series in \( r \).

Proof: \( H(x, r) := G(v(x), \lambda r^2) + \frac{r}{x} G(v(x), -\frac{\lambda}{\gamma} r^2) \). Thus, as formal power series,

\[
H(x, r) = \sum_{n=0}^{\infty} a_n P_n(v(x)) (\lambda r^2)^n + \frac{r}{x} \sum_{n=0}^{\infty} a_n P_n(v(x)) (\frac{\lambda}{\gamma} r^2)^n
\]

\[
= \sum_{n=0}^{\infty} \left( \lambda^n P_n(v(x)) \right) r^{2n} + \sum_{n=0}^{\infty} a_n \left( \left( -\frac{\lambda}{\gamma} \right)^n \frac{1}{x} P_n(v(x)) \right) r^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} b_{2n} \tilde{P}_{2n}(x) \, r^{2n} + \sum_{n=0}^{\infty} b_{2n+1} \tilde{P}_{2n+1}(x) \, r^{2n+1}
\]

\[
= \sum_{m=0}^{\infty} b_m \tilde{P}_m(x) \, r^m. \]

\( \square \)