CHAPTER 2
THE TRANSFORMATION

2.1. Definitions

For the remainder of this thesis, we assume that

\[ \psi \text{ is an MDF with monic } \text{OPS } \{P_n(x)\}_{n=0}^{\infty} \]

and that

\[ \lambda \text{ and } \gamma \text{ are fixed positive real numbers.} \]

Our development will be facilitated by a series of definitions: We set

\[ v(x) := \frac{1}{\lambda} \left( x - \frac{\gamma}{x} \right), \]

\[ v^{-1}_\pm(y) := \frac{\lambda}{2} \left( y \pm \sqrt{y^2 + \frac{4\gamma}{\lambda^2}} \right), \]

\[ \tilde{\psi}(x) := \begin{cases} \int_{-\infty}^{x} \frac{1}{v'(t)} d((\psi \circ v)(t)), & \text{for } x \in \mathbb{R}^- \\ \int_{0}^{x} \frac{1}{v'(t)} d((\psi \circ v)(t)), & \text{for } x \in \mathbb{R}^+ \end{cases}, \]

and, for \( n = 0, 1, 2, \ldots, \)

\[ \tilde{P}_{2n}(x) := \lambda^n P_n(v(x)) \text{ and } \tilde{P}_{2n+1}(x) := \left( -\frac{\lambda}{\gamma} \right)^n \frac{1}{x} P_n(v(x)). \]
2.2. Preliminary Theorems

We present here a progression of theorems, parts of which taken together constitute our main result, the Transformation Theorem, studied in the next section.

We begin with an exploration of $v$, the essential component of our transformation, and the related functions $v^{-1}_-, v^{-1}_+$, and $v'$.

**Theorem 2.2.1.**

(A) $v|_{\mathbb{R}^-}$, $v|_{\mathbb{R}^+}$, $v^{-1}_-$, and $v^{-1}_+$ are differentiable, monotone increasing functions.

(B) $v|_{\mathbb{R}^+}$ is a diffeomorphism from $\mathbb{R}^+$ to $\mathbb{R}$. Its inverse is $v^{-1}_+$. 

(C) $v|_{\mathbb{R}^-}$ is a diffeomorphism from $\mathbb{R}^-$ to $\mathbb{R}$. Its inverse is $v^{-1}_-$. 

(D) For all $x \in \mathbb{R}^- \cup \mathbb{R}^+$, $v(-\frac{\gamma}{x}) = v(x)$.

(E) For all $t \in \mathbb{R}^- \cup \mathbb{R}^+$, $\frac{dv}{dx}|_{x=-\frac{\gamma}{t}} = \frac{t^2}{\gamma} \frac{dv}{dx}|_{x=t}$.

Proof: (A) Evidently, $v$ has domain $\mathbb{R}^- \cup \mathbb{R}^+$, and $\frac{dv}{dx} = \frac{1}{x}(1 + \frac{\gamma}{x^2}) > \frac{1}{x} > 0$; that is, $v$ is differentiable and monotone increasing on $\mathbb{R}^-$ and on $\mathbb{R}^+$. Similarly, it is clear that the domain of $v^{-1}_\pm$ is $\mathbb{R}$, and, since

$$\left| \frac{y}{\sqrt{y^2 + \frac{4\gamma}{x^2}}} \right| < 1,$$
we have

\[
\frac{d v^{-1}_\pm}{dy} = \frac{\lambda}{2} \left(1 \pm \frac{y}{\sqrt{y^2 + \frac{4\gamma}{\lambda^2}}}ight) > 0;
\]

that is, \( v^{-1}_\pm \) is differentiable and monotone increasing. A graph of \( v, v^{-1}_-, \) and \( v^{-1}_+ \) is shown in Figure 1.

\[
y
y = v(x) \quad y = v^{-1}_+(x)
\]

\[
\sqrt{\gamma}
\]

\[
\sqrt{\gamma}
\]

\[
x
0
\]

\[
y = \frac{1}{\lambda} x \\
(asymptote)
\]

\[
y = v^{-1}_-(x) \quad y = v(x)
\]

\textit{Figure 1.} Illustration of basic features of \( v, v^{-1}_- \) and \( v^{-1}_+ \)

for typical values of \( \lambda \) and \( \gamma \).

(B) By inspection of the definitions, \( v|_{\mathbb{R}^+} \) maps \( \mathbb{R}^+ \) onto \( \mathbb{R} \), and \( v^{-1}_+ \) maps \( \mathbb{R} \) onto \( \mathbb{R}^+ \). By Theorem 2.2.1 (A), \( v|_{\mathbb{R}^+} \) and \( v^{-1}_+ \) are injective and differentiable. Hence, it suffices to show that \( v^{-1}_+(v(x)) = x \), for any \( x \) in
Theorem 2.2.1 (B).

(D) \( v(x) := \frac{1}{\lambda}(x - \gamma) = \frac{1}{\lambda}(\gamma - \gamma/(\gamma x)) = v(-\frac{\gamma}{x}) \), for any non-zero \( x \) in \( \mathbb{R} \).

(E) \( \frac{dv}{dx}\big|_{x=-\frac{\gamma}{t}} = \frac{1}{\lambda}(1 + \frac{\gamma^2}{x^2})|_{x=-\frac{\gamma}{t}} = \frac{1}{\lambda}(1 + \frac{t^2}{\gamma}) = \frac{t^2}{\gamma} \frac{dv}{dx}\big|_{x=t} \), for any non-zero \( t \) in \( \mathbb{R} \). □

Theorem 2.2.2.

(A) \( \tilde{\psi} \) is a bounded function on \( \mathbb{R}^- \cup \mathbb{R}^+ \).

(B) \( \tilde{\psi} \) is non-decreasing on \( \mathbb{R}^- \) and \( \mathbb{R}^+ \) separately.

Proof: (A) Inspection of the definitions shows that \( 0 < \frac{1}{v(t)} < \lambda \), for all non-zero \( t \) in \( \mathbb{R} \). Hence, by comparison, for \( x \) in \( \mathbb{R}^- \),
\[0 \leq \int_{-\infty}^{v(x)} \frac{1}{v'(v^{-1}(y))} \, d\psi(y) \leq \int_{-\infty}^{\infty} \lambda \, d\psi(y) = \lambda \mu_0(\psi).\]

But, for \(x\) in \(\mathbb{R}^-\), with \(y = v(t)\),

\[
\tilde{\psi}(x) := \int_{-\infty}^{x} \frac{1}{v'(t)} \, d(\psi \circ v)(t) = \int_{-\infty}^{v(x)} \frac{1}{v'(v^{-1}(y))} \, d\psi(y).
\]

Hence, \(0 \leq \tilde{\psi}(x) \leq \lambda \mu_0(\psi)\), for \(x\) in \(\mathbb{R}^-\); that is, \(\tilde{\psi}\) is a bounded map from \(\mathbb{R}^-\) to \(\mathbb{R}\). By a similar argument, one can show that \(\tilde{\psi}\) is a bounded map from \(\mathbb{R}^+\) to \(\mathbb{R}\).

(B) To verify that \(\tilde{\psi}\) is non-decreasing on \(\mathbb{R}^-\), suppose \(-\infty < x < y < 0\). For all \(t \leq y\), \(\frac{1}{v'(t)} \geq \frac{1}{v'(y)}\). Hence,

\[
\tilde{\psi}(y) - \tilde{\psi}(x) = \int_{x}^{y} \frac{1}{v'(t)} \, d(\psi \circ v)(t) \\
\geq \frac{1}{v'(y)} \int_{x}^{y} d(\psi \circ v)(t) \\
= \frac{\psi(v(y)) - \psi(v(x))}{v'(y)}.
\]

But, \(v'(y) = \frac{1}{\chi}(1 + \frac{\gamma}{y^2}) > 0\), and \(\psi(v(y)) - \psi(v(x)) \geq 0\) by the monotonicity of \(\psi\) and \(v\). Thus, \(\tilde{\psi}(y) - \tilde{\psi}(x) \geq 0\); that is, \(\tilde{\psi}\) is non-decreasing on \(\mathbb{R}^-\). Likewise, it can be shown that \(\tilde{\psi}\) is non-decreasing on \(\mathbb{R}^+\). \(\square\)

**Theorem 2.2.3.** \(\sigma(\tilde{\psi}) = v_-^{-1}(\sigma(\psi)) \cup v_+^{-1}(\sigma(\psi))\).

**Proof:** Suppose \(x \in v_-^{-1}(\sigma(\psi))\). Then \(x \in \mathbb{R}^-\), and \(v(x) \in v(v_-^{-1}(\sigma(\psi)))\)
= \sigma(\psi). Therefore, there is an \( \epsilon > 0 \) such that \((x - \epsilon, x + \epsilon) \subset \mathbb{R}^-\), and \( \psi(v(x) + \delta) - \psi(v(x) - \delta) > 0 \) for all \( \delta > 0 \). Let \( \delta_1 \) satisfy \( 0 < \delta_1 < \epsilon \). We have \( v(x - \delta_1) < v(x) < v(x + \delta_1) \) by Theorem 2.2.1 (A). Hence, there exists a \( \delta_2 > 0 \) such that \( v(x - \delta_1) < v(x) - \delta_2 < v(x) + \delta_2 < v(x + \delta_1) \), and the estimates

\[
\tilde{\psi}(x + \delta_1) - \tilde{\psi}(x - \delta_1) = \int_{v(x - \delta_1)}^{v(x + \delta_1)} \frac{1}{v'(\tilde{\psi}^{-1}(y))} d\tilde{\psi}(y)
\geq \frac{1}{v'(x + \delta_1)} \int_{v(x - \delta_1)}^{v(x + \delta_1)} d\tilde{\psi}(y)
= \frac{1}{v'(x + \delta_1)} (\psi(v(x + \delta_1)) - \psi(v(x - \delta_1)))
\geq \frac{1}{v'(x + \delta_1)} (\psi(v(x + \delta_2) - \psi(v(x) - \delta_2))
\]

hold. But \( \frac{1}{v'(x + \delta_1)} > 0 \), and, since \( v(x) \) is in \( \sigma(\psi) \), \( \psi(v(x) + \delta_2) - \psi(v(x) - \delta_2) > 0 \). Thus, \( \tilde{\psi}(x + \delta_1) - \tilde{\psi}(x - \delta_1) > 0 \). It follows that \( v_{-1}^{-1}(\sigma(\psi)) \subset \sigma(\tilde{\psi}) \). By a similar argument, one can show that \( v_{+1}^{-1}(\sigma(\psi)) \subset \sigma(\tilde{\psi}) \). Thus, \( (v_{-1}^{-1}(\sigma(\psi)) \cup v_{+1}^{-1}(\sigma(\psi))) \subset \sigma(\tilde{\psi}) \).

Next, suppose \( x \in (\sigma(\tilde{\psi}) \cap \mathbb{R}^-) \). By Theorem 2.2.1 (C), there is a unique \( y \) in \( \mathbb{R} \) such that \( v_{-1}^{-1}(y) = x \). But estimates similar to those above show that \( y \in \sigma(\psi) \). Thus, \( x \in v_{-1}^{-1}(\sigma(\psi)) \) if \( x \in (\sigma(\tilde{\psi}) \cap \mathbb{R}^-) \), and hence \( (\sigma(\tilde{\psi}) \cap \mathbb{R}^-) \subset v_{-1}^{-1}(\sigma(\psi)) \). In an analogous manner, it can be shown that \( (\sigma(\tilde{\psi}) \cap \mathbb{R}^+) \subset v_{+1}^{-1}(\sigma(\psi)) \). It follows that \( \sigma(\tilde{\psi}) = (\sigma(\tilde{\psi}) \cap \mathbb{R}^-) \cup (\sigma(\tilde{\psi}) \cap \mathbb{R}^+) \subset (v_{-1}^{-1}(\sigma(\psi)) \cup v_{+1}^{-1}(\sigma(\psi))) \). \( \square \)

**Theorem 2.2.4.** Let \( n \) be any integer. Then:
(A) $\int_{-\infty}^{0} x^n \frac{1}{v(x)} d(\psi \circ v)(x)$ and $\int_{0}^{\infty} x^n \frac{1}{v(x)} d(\psi \circ v)(x)$ exist.

(B) $\int_{-\infty}^{0} x^n \frac{1}{v(x)} d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} \frac{1}{v'(x)} d(\psi \circ v)(x)$.

(C) $\int_{0}^{\infty} x^n d\tilde{\psi}(x) = \int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)$ and $\int_{-\infty}^{0} x^n d\tilde{\psi}(x)$ exist.

(D) $\int_{0}^{\infty} x^n d\tilde{\psi}(x)$ and $\int_{0}^{\infty} x^n d\tilde{\psi}(x)$ exist.

(E) $\int_{0}^{\infty} x^n d\tilde{\psi}(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} d\tilde{\psi}(x)$.

Proof: (A) Since $0 < \frac{1}{v'(x)} < \lambda$, and, for all $x \in \mathbb{R}^-$, we have $0 < \frac{1}{v'(x)}$, we can deduce that the integrals $\int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)$ are finite by comparing $|t|^n$ to $t^N + 1$, for $N$ an even integer greater than $n$. Hence, for $n \in \mathbb{Z}_0^+$, the integrals $\int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)$ exist, by comparison. A similar argument shows that the integrals $\int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)$, for $n \in \mathbb{Z}_0^+$, exist.

The substitution $x \rightarrow -\frac{\gamma}{x}$ in $\int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x)$ yields, by Theorem...
2.2.1, parts (D) and (E),

\[
\int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} \frac{1}{v'(x)} d(\psi \circ v)(x).
\]

Hence, the integrals \( \int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \) and \( \int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \) exist for all integers \( n \), with the possible exception of the case \( n = -1 \). A comparison of \( |x|^{-1} \) to \( x^{-2} + 1 \) now shows that the integrals exist also for \( n = -1 \).

(B) By Theorem 2.2.4 (A), the integral \( \int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \) exists for any integer \( n \). As in the proof of Theorem 2.2.4 (A), the substitution \( x \to -\frac{\gamma}{x} \) now yields

\[
\int_{-\infty}^{0} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = (-1)^n \gamma^{n+1} \int_{0}^{\infty} x^{-n-2} \frac{1}{v'(x)} d(\psi \circ v)(x),
\]

for any integer \( n \).

(C, D) Let \( n \) be an integer, and suppose \( -\infty < a < b < 0 \). Then, since the integrands are continuous and the integrators are non-decreasing and bounded on the closed interval \( [a, b] \), the integrals

\[
\int_{a}^{b} x^n d\widetilde{\psi}(x) \quad \text{and} \quad \int_{a}^{b} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \quad \text{exist for all} \quad n \in \mathbb{Z}.
\]

Next, set \( x_{m,k} := k \frac{b-a}{m} + a \) for \( m \geq 1 \) and \( k = 0, 1, 2, ..., m \). By the Mean Value Theorem, there is a \( c_{m,k} \) in the closed interval \( [x_{m,k-1}, x_{m,k}] \) such that

\[
\int_{x_{m,k-1}}^{x_{m,k}} \frac{1}{v'(x)} d(\psi \circ v)(x) = \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1}))).
\]
for each \( k = 1, 2, \ldots, m \). Since the integrals exist, we can choose to take

\[
\int_a^b x^n d\tilde{\psi}(x) = \lim_{m \to \infty} \sum_{k=1}^m c_{m,k}^n (\tilde{\psi}(x_{m,k}) - \tilde{\psi}(x_{m,k-1}))
\]

and

\[
\int_a^b x^n \frac{1}{v'(x)} d(\psi \circ v)(x) = \lim_{m \to \infty} \sum_{k=1}^m c_{m,k}^n \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1}))).
\]

The definition of \( \tilde{\psi} \) and additivity of the integral imply

\[
\tilde{\psi}(x_{m,k}) - \tilde{\psi}(x_{m,k-1}) = \int_{x_{m,k-1}}^{x_{m,k}} \frac{1}{v'(x)} d(\psi \circ v)(x).
\]

It follows that

\[
\int_a^b x^n d\tilde{\psi}(x) = \lim_{m \to \infty} \sum_{k=1}^m c_{m,k}^n \frac{1}{v'(c_{m,k})} (\psi(v(x_{m,k})) - \psi(v(x_{m,k-1})))
\]

\[
= \int_a^b x^n \frac{1}{v'(x)} d(\psi \circ v)(x).
\]

Since \( \int_{-\infty}^0 x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \) exists by Theorem 2.2.4 (A), we then have \( \int_{-\infty}^0 x^n d\tilde{\psi}(x) \) exists and equals \( \int_{-\infty}^0 x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \).

Likewise, it follows that \( \int_{0}^{\infty} x^n d\tilde{\psi}(x) \) exists and is equal to the integral \( \int_{0}^{\infty} x^n \frac{1}{v'(x)} d(\psi \circ v)(x) \).

(E) The result follows by substitution of the integrals in Theorem 2.2.4 (C) in the equation of Theorem 2.2.4 (B). \( \square \)

**Theorem 2.2.5.** Let \( n \) be any integer. Then:

(A) \( \mu_n(\tilde{\psi}) \) exists.
(B) $\mu_n(\tilde{\psi}) = (-1)^n \gamma^{n+1} \mu_{-n-2}(\tilde{\psi})$.

(C) $\mu_{-1}(\tilde{\psi}) = 0$.

Proof: (A) Since $\mu_n(\tilde{\psi}) := \int_{-\infty}^{0} x^n d\tilde{\psi}(x) + \int_{0}^{\infty} x^n d\tilde{\psi}(x)$, Theorem 2.2.4 (D) implies $\mu_n(\tilde{\psi})$ exists for every integer $n$.

(B) The result follows immediately by applying Theorem 2.2.4 (E) to the definition $\mu_n(\tilde{\psi}) := \int_{-\infty}^{0} x^n d\tilde{\psi}(x) + \int_{0}^{\infty} x^n d\tilde{\psi}(x)$.

(C) By Theorem 2.2.5 (A), $\mu_{-1}(\tilde{\psi})$ exists, and, by Theorem 2.2.5 (B) with $n = -1$, $\mu_{-1}(\tilde{\psi}) = -\mu_{-1}(\tilde{\psi})$. Hence, $\mu_{-1}(\tilde{\psi}) = 0$. □

**Theorem 2.2.6.** $\tilde{\psi}$ is a SMDF.

Proof: By Theorem 2.2.2, $\tilde{\psi}$ is a bounded function, non-decreasing on $\mathbb{R}^-$ and $\mathbb{R}^+$ separately. By Theorem 2.2.1 (C), $v^{-1}_-$ and $v^{-1}_+$ are one-to-one, and Theorem 2.2.3 yields $\sigma(\tilde{\psi}) = v^{-1}_-(\sigma(\psi)) \cup v^{-1}_+(\sigma(\psi))$. Hence, $\sigma(\tilde{\psi})$ is infinite since $\sigma(\psi)$ is infinite, $\psi$ being a MDF. Lastly, the moment $\mu_n(\tilde{\psi})$, for each integer $n$, exists by Theorem 2.2.5 (A). □

**Theorem 2.2.7.** Let $R$ and $S$ be Laurent polynomials. Then the inner-product $(R, S)_{\tilde{\psi}} = \int_{-\infty}^{0} R(x)S(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \int_{0}^{\infty} R(x)S(x) \frac{1}{v'(x)} d(\psi \circ v)(x)$.

Proof: By Theorem 2.2.6, $\tilde{\psi}$ is a SMDF. Hence,

$$(R, S)_{\tilde{\psi}} = \int_{-\infty}^{0} R(x)S(x) d\tilde{\psi}(x) + \int_{0}^{\infty} R(x)S(x) d\tilde{\psi}(x).$$
Thus, the result follows from Theorem 2.2.4 (C) and linearity of the integral.

\[ \square \]

**Theorem 2.2.8.** Let \( j \) and \( k \) be non-negative integers. Then:

(A) \( (\tilde{P}_{2j}, \tilde{P}_{2k})_\psi = \lambda^{j+k+1}(P_j, P_k)_\psi \).

(B) \( (\tilde{P}_{2j+1}, \tilde{P}_{2k+1})_\psi = \left( \frac{\alpha}{\gamma} \right)^{j+k+1} (P_j, P_k)_\psi \).

(C) \( (\tilde{P}_{2j+1}, \tilde{P}_{2k})_\psi = 0 \).

Proof: (A) The definition of \((\tilde{P}_{2j}, \tilde{P}_{2k})_\psi\), Theorem 2.2.7, the substitution \(x = -\frac{\gamma}{t}\), Theorem 2.2.1 (E), linearity of the integral, the definition of \(\tilde{P}_{2j}(x)\) and \(\tilde{P}_{2k}(x)\), the substitution \(y = v(x)\), and the definition of \((P_j, P_k)_\psi\) justify

\[
(\tilde{P}_{2j}, \tilde{P}_{2k})_\psi := \int_{-\infty}^{0} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) d\tilde{\psi}(x) + \int_{0}^{\infty} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) d\tilde{\psi}(x)
\]

\[
= \int_{-\infty}^{0} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \int_{0}^{\infty} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x)
\]

\[
= \int_{0}^{\infty} \tilde{P}_{2j}(t)\tilde{P}_{2k}(t) \frac{\gamma}{t^2} \frac{1}{v'(t)} d(\psi \circ v)(t) + \int_{0}^{\infty} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) \frac{1}{v'(x)} d(\psi \circ v)(x)
\]

\[
= \lambda \int_{0}^{\infty} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) \frac{1}{\lambda(1 + \frac{\gamma}{x^2})} \frac{1}{v'(x)} d(\psi \circ v)(x)
\]

\[
= \lambda \int_{0}^{\infty} \tilde{P}_{2j}(x)\tilde{P}_{2k}(x) d(\psi \circ v)(x)
\]

\[
= \chi^{j+k+1} \int_{0}^{\infty} P_j(v(x))P_k(v(x)) d(\psi \circ v)(x)
\]

\[
= \chi^{j+k+1} \int_{-\infty}^{\infty} P_j(y)P_k(y) d\psi(y)
\]

\[
= \chi^{j+k+1}(P_j, P_k)_\psi.
\]
(B) By similar means as used in the proof of part (A),

\[
\begin{aligned}
\langle \tilde{P}_{2j+1}, \tilde{P}_{2k+1} \rangle_{\tilde{\psi}} := & \int_{-\infty}^{0} \tilde{P}_{2j+1}(x) \tilde{P}_{2k+1}(x) \, d\tilde{\psi}(x) + \\
& \int_{0}^{\infty} \tilde{P}_{2j+1}(x) \tilde{P}_{2k+1}(x) \, d\tilde{\psi}(x) \\
= & \left( -\frac{1}{\gamma} \right)^{j+k} \left( \int_{-\infty}^{0} \tilde{P}_{2j}(x) \tilde{P}_{2k}(x) \frac{1}{x^2} \, d\tilde{\psi}(x) + \\
& \int_{0}^{\infty} \tilde{P}_{2j}(x) \tilde{P}_{2k}(x) \frac{1}{x^2} \, d\tilde{\psi}(x) \right) \\
= & \left( -\frac{1}{\gamma} \right)^{j+k} \left( \int_{0}^{\infty} \tilde{P}_{2j}(t) \tilde{P}_{2k}(t) \frac{1}{x^2} \, d(\psi \circ v)(x) + \\
& \int_{0}^{\infty} \tilde{P}_{2j}(t) \tilde{P}_{2k}(t) \frac{1}{x^2} \, d(\psi \circ v)(x) \right) \\
= & \left( -\frac{1}{\gamma} \right)^{j+k} \frac{\lambda}{\gamma} \int_{0}^{\infty} \tilde{P}_{2j}(x) \tilde{P}_{2k}(x) \, d(\psi \circ v)(x) \\
= & \left( -\frac{1}{\gamma} \right)^{j+k} \frac{\lambda^{j+k+1}}{\gamma} \int_{-\infty}^{\infty} P_j(y) P_k(y) \, \psi(y) \\
= & \left( -1 \right)^{j+k} \left( \frac{\lambda}{\gamma} \right)^{j+k+1} (P_j, P_k)_{\psi}.
\end{aligned}
\]

If \( j \neq k \), then \( (P_j, P_k)_{\psi} = 0 \) by orthogonality. If \( j = k \), then \( (P_j, P_k)_{\psi} = 1 \).

In either case,

\[
\left( -1 \right)^{j+k} \left( \frac{\lambda}{\gamma} \right)^{j+k+1} (P_j, P_k)_{\psi} = \left( \frac{\lambda}{\gamma} \right)^{j+k+1} (P_j, P_k)_{\psi}.
\]

The result therefore follows.

(C) By arguments similar to those used in the proofs of the previous two
parts of Theorem 2.2.8,

\[
(\tilde{P}_{2j+1}, \tilde{P}_k)_{\tilde{\psi}} := \int_{-\infty}^{0} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \int_{0}^{\infty} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) d\tilde{\psi}(x)
\]

\[
= \int_{-\infty}^{0} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) \frac{1}{v'(x)} d(\psi \circ v)(x) + \int_{0}^{\infty} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) \frac{1}{v'(x)} d(\psi \circ v)(x)
\]

\[
= -\int_{0}^{\infty} \tilde{P}_{2j+1}(t) \tilde{P}_k(t) \frac{1}{v'(t)} d(\psi \circ v)(t) - \int_{-\infty}^{0} \tilde{P}_{2j+1}(t) \tilde{P}_k(t) \frac{1}{v'(t)} d(\psi \circ v)(t)
\]

\[
= -\int_{-\infty}^{0} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) d\tilde{\psi}(x) - \int_{0}^{\infty} \tilde{P}_{2j+1}(x) \tilde{P}_k(x) d\tilde{\psi}(x)
\]

\[
= -(\tilde{P}_{2j+1}, \tilde{P}_k)_{\tilde{\psi}}.
\]

Then, \((\tilde{P}_{2j+1}, \tilde{P}_k)_{\tilde{\psi}} = 0\), since \((\tilde{P}_{2j+1}, \tilde{P}_k)_{\tilde{\psi}}\) is finite by Theorem 2.2.6. □

**Theorem 2.2.9.** \{\(\tilde{P}_n(x)\)\}_{n=0}^{\infty} is the monic OLPS with respect to \(\tilde{\psi}\).

Proof: Inspection of the definition of \(\tilde{P}_n(x)\) shows that it is a monic L-polynomial of L-degree \(n\), and Theorem 2.2.8 implies orthogonality of \{\(\tilde{P}_n(x)\)\} \(_{n=0}^{\infty}\) with respect to \(\tilde{\psi}\). □

2.3. The Transformation Theorem

For ease of reference and discussion we collect several of the results obtained in the previous section into the following theorem.
**Theorem 2.3.1.** (The Transformation Theorem) Let \( \psi \) be a moment distribution function, let \( \sigma(\psi) \) denote the spectrum of \( \psi \), and let \( \{P_n(x)\}_{n=0}^{\infty} \) denote the monic orthogonal polynomial sequence with respect to \( \psi \). Let \( \lambda, \gamma \in \mathbb{R}^+ \), and set

\[
v(x) := \frac{1}{x}(x - \frac{\gamma}{x}) \quad \text{and} \quad v^{-1}(y) := \frac{\lambda}{2}(y \pm \sqrt{y^2 + \frac{4\gamma}{\lambda^2}}).
\]

Then:

(A) \( \tilde{\psi}(x) := \int_{-\infty}^{x} \frac{1}{v(t)} d(\psi \circ v)(t), \ x \in \mathbb{R}^- \) is a strong moment distribution function.

(B) \( \sigma(\tilde{\psi}) = v^{-1}(\sigma(\psi)) \cup v^{-1}(\sigma(\psi)) \) is the spectrum of \( \tilde{\psi} \).

(C) \( \{\tilde{P}_m(x)\}_{m=0}^{\infty} \) is the monic orthogonal Laurent polynomial sequence with respect to \( \tilde{\psi} \), where

\[
\tilde{P}_{2n}(x) := \lambda^n P_n(v(x)) \quad \text{and} \quad \tilde{P}_{2n+1}(x) := \left( -\frac{\lambda}{\gamma} \right)^n \frac{1}{x} P_n(v(x))
\]

for \( n = 0, 1, 2, \ldots \).

Proof: See the proofs of Theorem 2.2.3, Theorem 2.2.6 and Theorem 2.2.9 \( \square \)

We call \( v \) the **doubling transformation** because it is a monotone increasing function of both \( \mathbb{R}^- \) and \( \mathbb{R}^+ \) onto \( \mathbb{R} \). In effect, \( (f \circ v)|_{\mathbb{R}^-} \) and \( (f \circ v)|_{\mathbb{R}^+} \) are copies of \( f : \mathbb{R} \to \mathbb{R} \) living on the negative reals and the positive reals, respectively. In this sense, \( f \circ v \) is a doubling of \( f \). See Figure 2 for an example.
Figure 2. Graph of the monic Legendre polynomial $P_3(x) = x^3 - \frac{3}{5}x$ and $\tilde{P}_6(x) = \lambda^3 P_3(v(x))$ with $\lambda = \gamma = 1$.

Of course, $v$ is not the only monotone increasing function of both $\mathbb{R}^-$ and $\mathbb{R}^+$ onto $\mathbb{R}$; that is, $v$ is not the only doubling transformation. However, $v(x) = \frac{1}{\lambda}(x - \frac{2}{x})$ is a Laurent polynomial. This feature, along with those given in Theorem 2.2.1, make $v$ especially useful for the purpose of transforming systems of OPS’s into systems of OLPS’s. Inspection of the Transformation Theorem shows that the L-polynomial $\tilde{P}_{2n}$ is a doubling of the polynomial $P_n$, and, in a slightly looser sense, $\bar{\psi}$ with spectrum $\sigma(\bar{\psi}) = v^{-1}(\sigma(\psi)) \cup v^{-1}_+(\sigma(\psi))$ is a doubling of $\psi$ with spectrum $\sigma(\psi)$.

The doubling of the spectrum in particular can be used to discuss to
what extent $\tilde{\psi}$ is an extension of $\psi$. For example, if $\sigma(\psi)$ is a symmetric set about the origin, it can be seen by Theorem 2.3.1 (B) and the definitions of $v_{-}^{-1}$ and $v_{+}^{-1}$ that $\sigma(\tilde{\psi})$ is symmetric about the origin. When $\sigma(\psi)$ is a symmetric interval about the origin, $\sigma(\tilde{\psi})$ is the union of two disjoint intervals forming a set symmetric about the origin. In particular, if $\sigma(\psi) = \mathbb{R}$, then $\sigma(\tilde{\psi}) = \mathbb{R}^{-} \cup \mathbb{R}^{+}$. If $\sigma(\psi) \subseteq \mathbb{R}_{0}^{+} = [0, \infty)$, we would like an extension of $\psi$ to a SMDF to have its spectrum contained in $\mathbb{R}^{+}$. However, a direct application of the Transformation Theorem to a MDF $\psi$ having spectrum $\sigma(\psi) \subseteq \mathbb{R}_{0}^{+}$ yields the SMDF $\tilde{\psi}$ with its doubled spectrum $\sigma(\tilde{\psi}) = v_{-}^{-1}(\sigma(\psi)) \cup v_{+}^{-1}(\sigma(\psi))$ half contained in $\mathbb{R}^{-}$. Similarly, if $\sigma(\psi) \subseteq \mathbb{R}_{0}^{-} = (-\infty, 0]$, then $\sigma(\tilde{\psi})$ is half contained in $\mathbb{R}^{+}$.

In a further effort to discover to what extent the transformed objects given by the Transformation Theorem are extensions of the corresponding original objects, it is worth examining the limiting case of $\lambda = 1$ and $\gamma = 0$. In this situation, it can be seen by inspection of the definitions that $v(x) = x$ and $v_{\pm}^{-1}(x) = xI_{\mathbb{R}_{\pm}}(x)$, where $I_{A}(x)$ is the indicator function for a set $A$. Hence, in this limiting case, we see that $\tilde{P}_{2n}(x) = P_{n}(x)$, $v_{\pm}^{-1}(\sigma(\psi)) = \sigma(\psi) \cap \mathbb{R}_{0}^{\pm}$ and $\sigma(\tilde{\psi}) = \sigma(\psi)$. Further, as we will show in the next chapter, $\tilde{\psi}(x)$ and $\psi(x)$ are essentially equal in this limiting case, with $\mu_{n}(\tilde{\psi}) = \mu_{n}(\psi)$ for all non-negative integers $n$. 