OPTIMIZING GAUSSIAN QUADRATURE

ABSTRACT. Classical Gaussian quadrature is anchored in the space of real polynomials at polynomial degree 0 and is monotonically directed by successively increasing polynomial degree. The results of recent research, investigating weighing anchor and charting various courses in search of the best currents in spaces of real Laurent polynomials, are presented. Featured are standard error bound minimization anchoring and steering algorithms.

REFERENCES


FURTHER READING


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This PDF file resides online at
http://general.utpb.edu/FAC/hagler_b/Research/Talks/OGQ/talk.pdf
Theorem 0.1. ([1] Theorem 3.13) Let \( \langle \{z^{p(n)}\}_{n=1}^{k}\rangle \) be the Laurent polynomial space over \( \mathbb{F} \) with order implied by ordered basis \( A = \{z^{p(n)}\}_{n=1}^{k} \) for some \( k \in \mathbb{N} \cup [\infty] \) and injection \( p : [n \in \mathbb{N} : 1 \leq n < k + 1] \rightarrow \mathbb{Z} \). Let \( \mathcal{L} : \Omega \rightarrow \mathbb{C} \) be a strong moment functional, moments \( \mu_{n} = \mathcal{L}(z^{n}) \), \( n \in \mathbb{Z} \). Then:

(A) \( R \cdot S = \mathcal{L}(RS) \) is an inner product on \( \langle \{z^{p(n)}\}_{n=1}^{k}\rangle \) over \( \mathbb{F} \) if and only if

\[
(0.1a) \quad \mathbb{F} \text{ is a subfield of } \mathbb{R},
\]

\[
(0.1b) \quad \mu_{p(i)+p(j)} \in \mathbb{F} \text{ for all } i, j \text{ such that } 1 \leq i, j < k + 1,
\]

and

\[
(0.1c) \quad |G_{m}| := \begin{vmatrix}
\mu_{p(1)+p(1)} & \mu_{p(1)+p(2)} & \cdots & \mu_{p(1)+p(m)} \\
\mu_{p(2)+p(1)} & \mu_{p(2)+p(2)} & \cdots & \mu_{p(2)+p(m)} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{p(m)+p(1)} & \mu_{p(m)+p(2)} & \cdots & \mu_{p(m)+p(m)}
\end{vmatrix} > 0.
\]

(B) If \( \mathbb{F} = \mathbb{R} \) and \( p \) is a bijection from \( \mathbb{N} \) to \( \mathbb{Z} \), then \( \langle \{z^{p(n)}\}_{n=1}^{k}\rangle \) is the space \( \Lambda^{\mathbb{R}} := \left\{ \sum_{i=m}^{n} \tau_{i}x^{i} : m, n \in \mathbb{Z}, \tau_{i} \in \mathbb{R} \text{ for } m \leq i \leq n \right\} \) of all real Laurent polynomials, and the following conditions are equivalent:

(I) \( \mathcal{L}(S^{2}) > 0 \) for all non-zero \( S \) in \( \Lambda^{\mathbb{R}} \).

(II) \( (0.1c) \) with \( k = \infty \) holds.

(III) \( \mathcal{L}(R) > 0 \) for all non-zero \( R \) in \( \Lambda^{\mathbb{R}} \) such that \( R(x) \geq 0 \) for all \( x \in \mathbb{R}^{*} := \mathbb{R} - [0] \) (i.e., \( \mathcal{L} \) is positive definite).

Furthermore, (I)-(III) are implied by the condition

\( \text{(SMDF)} \quad \mathcal{L}(Q) = \int_{\mathbb{R}^{*}} Q(x) d\phi(x) \) for some \( \phi \in \Phi \) and all \( Q \) in \( \Lambda^{\mathbb{R}} \).

(C) If \( (0.1) \) holds, then \( \{Q_{n}(x)\}_{n=1}^{k} \) such that \( |G_{0}| := 1 \) and

\[
Q_{n}(x) = |G_{n-1}|^{-1} \begin{vmatrix}
\mu_{p(1)+p(1)} & \cdots & \mu_{p(1)+p(n-1)} & x^{p(1)} \\
\vdots & \ddots & \vdots & \vdots \\
\mu_{p(n)+p(1)} & \cdots & \mu_{p(n)+p(n-1)} & x^{p(n)}
\end{vmatrix}
\]

is the monic OLPS in \( \langle \{x^{p(n)}\}_{n=1}^{k}\rangle \) with respect to the inner product \( R \cdot S = \mathcal{L}(RS) \) and ordered basis \( A = \{x^{p(n)}\}_{n=1}^{k} \).
Theorem 0.2. (Theorem 5.1) Let \( \{A_n\}_{n=1}^k \) be an OLPS with respect to an inner product \( \cdot \) on an \( L \) space \( V \). Let rank \( \rho \) and leading coefficient \( \kappa \) of elements of \( V \) be relative to \( \{A_n\}_{n=1}^k \). Then:

\( \{A_n\}_{n=1}^k \) satisfies the system of three-term recurrence relations

\[ (0.3a) \quad A_{-1} := 1, \quad A_0 := 0, \quad A_n = b_n A_{n-1} + a_n A_{n-2} \text{ for } 1 \leq n < k + 1 \]

where

\[ (0.3b) \quad b_n := q_n - \frac{(q_n A_{n-1}) \cdot A_{n-1}}{A_{n-1} \cdot A_{n-1}} \text{ for } n \geq 2, \]

\[ (0.3c) \quad a_1 := A_1, \quad a_n := -\frac{(q_n A_{n-1}) \cdot A_{n-2}}{A_{n-2} \cdot A_{n-2}} \neq 0 \text{ for } n \geq 3 \]

if and only if, for all \( n \) such that \( 2 \leq n < k + 1 \), \( \rho(q_n A_{n-1}) = n \), \( \kappa(q_n A_{n-1}) = 1 \), \( (q_n A_{n-1}) \cdot A_{n-2} \neq 0 \) if \( n \geq 3 \), and \( (q_n A_{n-1}) \cdot A_j = 0 \) if \( 1 \leq j \leq n - 3 \).
### Features of Block-Formed OLPS Examples for a Hermite PDSMF.

**Example 1.** $A = \{1, x, x^2, x^3, x^4, x^5, x^6\}$  
$L[r(x)] = \int_{-\infty}^{\infty} r(x) e^{-(x-1/2)^2} \, dx$

<table>
<thead>
<tr>
<th>Rank</th>
<th>$\rho$, Monic OLPS Member</th>
<th>$p(A, \rho)$</th>
<th>$Q_\rho(x)/x^{p(A, \rho)}$</th>
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<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>$x$</td>
<td>0</td>
<td>$x$</td>
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<tr>
<td>3</td>
<td>$-\frac{3}{4} + x^2$</td>
<td>0</td>
<td>$-\frac{3}{4} + x^2$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{13}{16} + x^3$</td>
<td>0</td>
<td>$-\frac{13}{16} + x^3$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{3}{8} - \frac{19}{4} x^2 + x^4$</td>
<td>0</td>
<td>$\frac{3}{8} - \frac{19}{4} x^2 + x^4$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{877}{124} x^2 - \frac{193}{32} x^4 + x^5$</td>
<td>0</td>
<td>$\frac{877}{124} x^2 - \frac{193}{32} x^4 + x^5$</td>
</tr>
<tr>
<td>7</td>
<td>$-\frac{865}{61} + \frac{5967}{244} x^2 - \frac{603}{61} x^4 + x^6$</td>
<td>0</td>
<td>$-\frac{865}{61} + \frac{5967}{244} x^2 - \frac{603}{61} x^4 + x^6$</td>
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</table>

**Example 2.** $A = \{1, \frac{1}{2}, x, \frac{1}{2} x, x^2, \frac{1}{2} x^2, x^3\}$

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<th>Rank</th>
<th>$\rho$, Monic OLPS Member</th>
<th>$p(A, \rho)$</th>
<th>$Q_\rho(x)/x^{p(A, \rho)}$</th>
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<td>1</td>
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<td>$-1 + x^2$</td>
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<tr>
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<td>$\frac{1}{2} x - \frac{1}{2}$</td>
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<td>$1 - \frac{5}{2} x^2 + x^4$</td>
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<tr>
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<td>$-2$</td>
<td>$1 - \frac{9 x^2}{2} - \frac{9 x^4}{2} + x^6$</td>
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**Example 3.** $A = \{\frac{1}{2}, \frac{1}{2} x, 1, \frac{1}{2} x, x, \frac{1}{2} x, x^2\}$

<table>
<thead>
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<th>Rank</th>
<th>$\rho$, Monic OLPS Member</th>
<th>$p(A, \rho)$</th>
<th>$Q_\rho(x)/x^{p(A, \rho)}$</th>
</tr>
</thead>
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<td>$\frac{1}{2}$</td>
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</tr>
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<td>2</td>
<td>$\frac{1}{2} x$</td>
<td>$-1$</td>
<td>$\frac{1}{2}$</td>
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<td>$-\frac{2}{32} + 1$</td>
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<td>$-\frac{2}{3} + x^2$</td>
</tr>
<tr>
<td>4</td>
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<td>$\frac{1}{2} x - \frac{3}{2} x^2$</td>
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<td>$\frac{1}{2} x - \frac{7}{22} x^2 + 2$</td>
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<td>$\frac{1}{2} x - \frac{7}{22} x + x^5$</td>
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<td>1</td>
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</tr>
<tr>
<td>8</td>
<td>$\frac{1}{2} x - \frac{39}{224} x + \frac{291}{224} - \frac{61}{16}$</td>
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<td>$\frac{1}{2} x - \frac{39}{224} x + \frac{291}{224} - \frac{61}{16}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{6}{547} - \frac{68}{547} x + \frac{429}{1062} x - \frac{423}{106} + x^2$</td>
<td>$-6$</td>
<td>$\frac{6}{547} - \frac{68}{547} x + \frac{429}{1062} x - \frac{423}{106} + x^8$</td>
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</table>
Lemma 0.3. If \( \{Q_n(x)\}_{n=1}^{k} \) for some \( k \in \mathbb{N} \cup [\infty] \) is an OLPS with respect to a PDSMF and ordered basis of the form \( \{x^{p+(n-1)}\}_{n=1}^{k} \) or \( \{x^{p-(n-1)}\}_{n=1}^{k} \) for some \( p \in \mathbb{Z} \), then \( Q_n(x)/x^p \) has \( n-1 \) simple zeros in \( \mathbb{R} \cup [\infty] \) for each index \( n \); i.e., \( \{Q_n(x)\}_{n=1}^{k} \) is strongly regular.

Proof. Without loss of generality, assume \( \{Q_n(x)\}_{n=1}^{k} \) is monic. Take \( q_n = x \) in the case of ordered basis \( \{x^{p+(n-1)}\}_{n=1}^{k} \) or \( q_n = 1/x \) in the case of \( \{x^{p-(n-1)}\}_{n=1}^{k} \), and apply Theorem 0.2 to show \( \{Q_n(z)\}_{n=1}^{k} \) satisfies a system of three-term recurrence of the form (0.3). Hence, \( \{Q_n(x)/x^p\}_{n=1}^{k} \) also satisfies that system, and the canonical proof for orthogonal polynomial sequences (the case with ordered basis \( \{1, x, x^2, \ldots\} \)) can be adapted to finish the proof. \( \square \)

Remark 0.4. To apply Theorem 0.2 in the previous proof, one needs to verify \( (q_n Q_{n-1}) \cdot Q_{n-2} \neq 0 \) if \( n \geq 3 \) and \( (q_n Q_{n-1}) \cdot Q_j = 0 \) if \( 1 \leq j \leq n-3 \), both of which depend on the form of the inner product; in particular, for L-polynomials \( R \) and \( S \), \( (q_n R) \cdot S = L[(q_n R)S] = L[R(q_n S)] = R \cdot (q_n S) \), and \( (q_n Q_{n-1}) \cdot Q_{n-2} = Q_{n-1} \cdot (q_n Q_{n-2}) = Q_{n-1} \cdot Q_{n-1} = L[Q_{n-1}^2] > 0 \) since \( L \) is positive definite.

Proposition 0.5. Every block-formed OLPS with respect to a PDSMF is strongly regular.
Special Block-formed Quadrature Error for a Hermite PDSMF $\mathcal{L}$ and an Example Block-formed Basis $A$.

$$\mathcal{L}[r(x)] = \int_{-\infty}^{\infty} r(x) e^{-(x-1/x)^2} \, dx \quad A = \{1/x, 1/x^2, 1/x^3, 1, x, 1/x^4\}$$

<table>
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<tr>
<th>$p$</th>
<th>$E_2[x^p]$</th>
<th>$E_3[x^p]$</th>
<th>$E_4[x^p]$</th>
<th>$E_5[x^p]$</th>
<th>$E_6[x^p]$</th>
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Special Block-formed Quadrature Error

(0.4) \[ E_n[r(x)] := \mathcal{L}[r(x)] - \sum_{j=1}^{n-1} \left[ \frac{r(x)}{x^{2p(A,n)}} \right]_{x=x_{n,j}} w_{n,j}, \]

where $x_{n,j}$ are the simple zeros of $Q_n(x)/x^{p(A,n)}$

and $w_{n,j} := \mathcal{L} \left[ x^{2p(A,n)} \frac{P_{n-1}(x^{d(n)})}{(x^{d(n)} - x_{n,j}^{d(n)}) P'_{n-1}(x_{n,j}^{d(n)})} \right]$, $j = 1, \ldots, n-1$,

for $P_{n-1}(x) = \prod_{j=1}^{n-1} (x - x_{n,j}^{d(n)})$.

**Key:**
- **error numbers** Numerically derived values using Formula (0.4)
- **[medium gray block]** The exact power block at rank $n$ in $A$
- **[light gray block]** The strongly regular power ray, without the exact power block, at rank $n$ in $A$
- **---** $E_n[x^p]$ isn’t given by Formula (0.4)
Power Blocks for an Example Block-formed Basis $A$.

$$A = \{1/x, 1/x^2, 1/x^3, 1, x, 1/x^4\}$$

Key: Top to bottom, the graphs correspond to rank $n$ in $A$, from 1 to 6.
- [black box] The power $p(n)$
- [dark gray block] The set $[p(1), \ldots, p(n-1)]$
- [medium gray block] The exact power block at rank $n$ in $A$
- [light gray block] The strongly regular power ray, without the exact power block, at rank $n$ in $A$
### Block-formed Quadrature Error for a Legendre PDSMF $L$ and an Example Block-formed Basis $A$.

$$L[r(x)] = \int_{1}^{2} r(x) \, dx \quad A = \{1/x, 1/x^2, 1/x^3, 1, x, 1/x^4\}$$

### Table of Block-formed Quadrature Error

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<thead>
<tr>
<th>$p$</th>
<th>$E_1[x^p]$</th>
<th>$E_2[x^p]$</th>
<th>$E_3[x^p]$</th>
<th>$E_4[x^p]$</th>
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### Key
- **error numbers**
  - Numerical values, Formula (0.4) or (0.5)
- **medium gray block**
  - The exact power block at rank $n$ in $A$
- **light gray block**
  - The strongly regular power ray, without the exact power block, at rank $n$ in $A$
Block-formed Quadrature Error for a Hermite PDSMF $L$ and an Example Block-formed Basis $A$.

$$L[r(x)] = \int_{-\infty}^{\infty} r(x) e^{-(x-l-x)^2} \, dx \quad A = \{1/x, 1/x^2, 1/x^3, 1, x, 1/x^4\}$$

For each natural number $n$,

$$L[r(x)] = \sum_{i=1}^{\nu_n} r(t_{n,i})\lambda_{n,i} \text{ for all } r(x) \in \langle \{x^{2p(A,n)+d(n)\times(i-1)}\}^{2n-2}_{i=1} \rangle_{i=n-\nu_n}.$$

**Key:**
- **error numbers** Numerically derived values using Formula (0.5)
- **[medium gray block]** The exact power block for special Quadrature with 0 error by Formula (0.4)

<table>
<thead>
<tr>
<th>p</th>
<th>$\mathcal{E}_1[x^p]$</th>
<th>$\mathcal{E}_2[x^p]$</th>
<th>$\mathcal{E}_3[x^p]$</th>
<th>$\mathcal{E}_4[x^p]$</th>
<th>$\mathcal{E}_5[x^p]$</th>
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For each natural number $n$, 

$$L[r(x)] = \sum_{i=1}^{\nu_n} r(t_{n,i})\lambda_{n,i} \text{ for all } r(x) \in \langle \{x^{2p(A,n)+d(n)\times(i-1)}\}^{2n-2}_{i=1} \rangle_{i=n-\nu_n}.$$
Proposition 0.6. Suppose $L$ is a PDSMF of the form

$$L[f(x)] = \int_{a}^{b} f(x) \, w(x) \, dx,$$

corresponding to a continuous, non-negative weight function $w(x)$ on an interval $[a, b] \subset \mathbb{R}^*$. Then, for each rank $n$, any block-formed orthogonal Laurent polynomial $Q_n(x)$ has $\nu_n = n - 1$ simple zeros in the open interval $(a, b)$.

Proof. $Q_1(x) = x^{p(1)}$ has no zeros in $(a, b)$ since this interval does not contain 0 or $\infty$. Now, let rank $n$ be greater than 1, and adapt the canonical path to proof for the polynomial case as follows. By orthogonality, $L[x^{p(A,n)} Q_n(x)] = 0$, which implies $Q_n(x)$ must change sign at least once in $(a, b)$, hence $Q_n(x)$ has at least one zero of odd multiplicity in $(a, b)$. Let $q(x) = \prod_{i=1}^{j} (x - x_i^{d(n)})$ where $x_i$ for $1 \leq i \leq j$ denote the distinct zeros of odd multiplicity of $Q_n(x)$ in $(a, b)$. Then $x^{p(A,n)} q(x^{d(n)}) Q_n(x)$ is an L-polynomial that has no zeros in $(a, b)$ of odd multiplicity, hence one finds $x^{p(A,n)} q(x^{d(n)}) Q_n(x) \geq 0$ for all $x$ in $(a, b)$. It follows that $L[x^{p(A,n)} q(x^{d(n)}) Q_n(x)] > 0$. If the degree $j$ of $q(x)$ is less than $n - 1$, then $x^{p(A,n)} q(x^{d(n)})$ is in the linear span of $\{x^{p(i)}\}_{i=1}^{n-1}$, and it would follow that $L[x^{p(A,n)} q(x^{d(n)}) Q_n(x)]$ is zero, contradicting $L[x^{p(A,n)} q(x^{d(n)}) Q_n(x)] > 0$. Hence, $j \geq n - 1$. But $Q_n(x)$ has at most $n - 1$ distinct zeros in $\mathbb{R}^*$ in total, hence $j = n - 1$, which completes the proof. \qed
Quadratures $Q_n[r(x)]$ and Corresponding Standard Error Bounds

$\mathcal{B}_n[r(x)]$, Varying by Block-formed Basis $A$.

$$\mathcal{L}[r(x)] = \int_a^b r(x) \, dx \quad r(x) = 1$$

$$\mathcal{B}_n[r(x)] := \frac{1}{(2(n-1))!} \left( \max_{\xi \in [a,b]} |R^{(2(n-1))}(\xi)| \right) \int_a^b Q_n^2(x) \, dx$$

where $R(x) := r(x^{d(n)})/x^{2(d(n) \times p(n,n))}$ is in $C^{2(n-1)}[a,b]$.

<table>
<thead>
<tr>
<th>Basis $A$</th>
<th>$Q_2[x(x)]$</th>
<th>$B_2[x(x)]$</th>
<th>$Q_5[x(x)]$</th>
<th>$B_5[x(x)]$</th>
</tr>
</thead>
<tbody>
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<td>${x^3, x^4, x^5, x^6, x}$</td>
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<td>0.997564</td>
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<td>${x, x^2, x^3, x^4, x}$</td>
<td>0.607098</td>
<td>1.95565</td>
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<td>${x, x^2, x^3, x^4, 1}$</td>
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</tr>
</tbody>
</table>

Given a PDSMF $\mathcal{L}$ and a function $r(x)$ such that $\mathcal{L}[r(x)]$ exists, which choice of initial power $p(1) \in \mathbb{Z}$ and directions $d(i) \in [\pm 1]$, for ranks $i = 2, \ldots, n$, gives the Optimal Gaussian Quadrature performance, minimizing error magnitude $|E_n[r(x)]|$?

In using the standard error bound to gauge which error is least, a smaller bound doesn’t necessarily mean a smaller actual magnitude of error. The ideal initialization is a selection of $p(1)$ and $d(2)$ such that the corresponding $|E_2[r(x)]|$ is as small as possible, but $\mathcal{L}[r(x)]$ is presumably unknown, hence $E_2[r(x)]$ cannot be determined exactly. Even if the value of $\mathcal{L}[r(x)]$ can be computed, there are no guarantees that such ideal $p(1)$ and $d(2)$ exist, or, if they exist, that they are unique. And ultimately, if they exist and are unique, they may not lead to the best results at the final rank; i.e., there may be a different initial choice of $p(1)$ and $d(2)$ whose $|E_2[r(x)]|$ is larger, but whose $|E_n[r(x)]|$ is smaller after following a sequence of directions $d(3), \ldots, d(n)$.

Having noted some of the inherent difficulties, there are obvious advantages to be gained in using error bounds, or other information, ignored by random, polynomial or the usual Laurent polynomial block-formed quadratures.
Anchoring and Steering Algorithms.

INPUT (1) The maximum allowable magnitude $MaxAbsPower$ of the initial power. (2) The function $r(x)$ being analyzed. (3) Moments $\mu_k$.

OUTPUT An initial power $p(1)$ and direction $d(2)$ which, in a neighborhood of 0, minimizes the rank 2 Standard Error Bound.

Step 1 Set $p_1 = 0$ and $g = 1$, and define the measure

$$B_p = \max_{x \in [a,b]} \left| \frac{d^2}{dx^2} \left( \frac{r(x)}{x^2} \right) \right| \times (\mu_{2p+2} - \mu_{2p+1}/\mu_{2p}) + \max_{x \in [a,b]} \left| \frac{d^2}{dx^2} \left( x^2p \frac{r(1/x)}{x} \right) \right| \times (\mu_{2p-2} - \mu_{2p-1}/\mu_{2p})$$

Step 2 If $|p_1| \geq MaxAbsPower$, OUTPUT (Message: Power range restriction reached.) and go to Step 5.

Step 3 If $B_{p_1+g} < B_{p_1}$, set $p_1 = p_1 + g$ and go to Step 2.

Step 4 If $B_{p_1+g} \geq B_{p_1}$ and $p_1 = 0$ and $g \neq -1$, set $g = -1$ and go to Step 3.

Step 5 Set $p(1) = p_1$ and OUTPUT($p_1$).

STOP.

The Gram matrix determinants are

$$|G_m| = \begin{vmatrix} \mu_{p(1)+p(1)} & \mu_{p(1)+p(2)} & \cdots & \mu_{p(1)+p(m)} \\ \mu_{p(2)+p(1)} & \mu_{p(2)+p(2)} & \cdots & \mu_{p(2)+p(m)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{p(m)+p(1)} & \mu_{p(m)+p(2)} & \cdots & \mu_{p(m)+p(m)} \end{vmatrix}$$

If we have initialized $p(1)$ and established $d(2), \ldots, d(n-1)$, it suffices to choose $d(n)$ by comparing the measures

(0.6a) $\max_{x \in [a,b]} \left| \frac{d^{2(n-1)}}{dx^{2(n-1)}} \left( \frac{r(x)}{x^{2m(n-1)}} \right) \right| \times |G_{n[p(n)=M(n-1)+1}}|

and

(0.6b) $\max_{x \in [a,b]} \left| \frac{d^{2(n-1)}}{dx^{2(n-1)}} \left( \frac{r(1/x)}{x^{2M(n-1)}} \right) \right| \times |G_{n[p(n)=m(n-1)-1}}|

to minimize the standard error bound at rank $n$; if the first is smaller than the second, take $d(n) = 1$ (i.e., $p(n) = M(n-1) + 1$), if the second is smaller, take $d(n) = -1$ (i.e., $p(n) = m(n-1) - 1$), and if the two measures are equal, we will take $d(n) = d(n-1)$. 

### Block-formed Quadrature Comparisons.

<table>
<thead>
<tr>
<th>Method</th>
<th>n</th>
<th>Basis</th>
<th>(Q_n[r(x)])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>Poly 4</td>
<td>(\left{1, x, x^2, x^3\right})</td>
</tr>
<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>Poly 5</td>
<td>(\left{1, x, x^2, x^3, x^4\right})</td>
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<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>L-Poly 4</td>
<td>(\left{1, \frac{1}{x}, x, \frac{1}{x^2}\right})</td>
</tr>
<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>L-Poly 5</td>
<td>(\left{1, \frac{1}{x}, x, \frac{1}{x^2}, x^2\right})</td>
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<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>SEBM 4</td>
<td>(\left{x, 1, x^2, \frac{1}{x}\right})</td>
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<tr>
<td>(\sqrt{x})</td>
<td>(\int_{1/4}^{1} e^{-x} r(x) , dx)</td>
<td>SEBM 5</td>
<td>(\left{x, 1, x^2, \frac{1}{x}, x^3\right})</td>
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<tr>
<td>(x^7)</td>
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<td>Poly 4</td>
<td>(\left{1, x, x^2, x^3\right})</td>
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