Selected Homework Solution Sketches

4.14. (a) \( a + b \leq \sup A + b \leq \sup A + \sup B \), for all \( a \in A \) and \( b \in B \), shows \( \sup A + \sup B \) is an upper bound for \( S \). Proceed by contradiction. Assume there exists a real number \( M \) such that \( a + b \leq M < \sup A + \sup B \), for all \( a \in A \) and \( b \in B \). Then \( a \leq M - b \), for all \( a \in A \) and \( b \in B \); i.e., \( M - b \) is an upper bound for \( A \). One can deduce that \( a \leq \sup A \leq M - b \), for all \( a \in A \) and \( b \in B \), since \( \sup A \) is the least upper bound. Thus, \( a + b \leq \sup A + b \leq M < \sup A + \sup B \), for all \( a \in A \) and \( b \in B \). But we then have \( b \leq M - \sup A < \sup B \), for all \( b \in B \). \( \to \) This contradicts \( \sup B \) being the least upper bound for \( B \).

(b) Applying Exercises 4.9 and 4.14(a),
\[
\inf S = -\sup(-S) = -((-\sup(-A) + \sup(-B)) = (-\sup(-A)) + (-\sup(-B)) = \inf A + \inf B.
\]

8.4. Let \( \varepsilon > 0 \). Suppose \( (t_n) \) is bounded; i.e., \( |t_n| \leq M \) for all \( n \). Assume \( (s_n) \) is a sequence such that \( \lim s_n = 0 \). There exists \( N \) such that
\[
n > N \implies |s_n - 0| < \frac{\varepsilon}{M}, \text{ or } |s_n| < \frac{\varepsilon}{M},
\]
since \( \lim s_n = 0 \). Also recall that
\[
|t_n| \leq M.
\]
Now, \( n > N \) implies
\[
|t_n s_n - 0| = |t_n| |s_n| < \frac{\varepsilon}{M} = \varepsilon.
\]
Hence, \( \lim s_n t_n = 0 \).

10.10. (b) Key: If \( s_n > 1/2 \), then \( s_{n+1} = (s_n + 1) / 3 > (1/2 + 1) / 3 = 1/2 \).
(c) Key: \( s_{n+1} - s_n = (s_n + 1) / 3 - s_n = 1/3 - s < 1/3 - 1/3 = 0 \)
(d) (b) and (c) imply \( (s_n) \) converges, by Monotone Convergence (10.2). Now, proceeding as we’ve seen in Ex. 9.4, we can calculate \( \lim s_n = 1/2 \).
10.12. (a) Show $0 < t_{n+1} < t_n$ for all $n$, and invoke the Monotone Convergence Theorem.

(c) $t_{k+1} = \left[ \frac{1}{(k+1)^2} \right] - \left[ \frac{1}{k} \right] t_k = \left[ \frac{1}{(k+1)^2} \right] (k+1) = \frac{(k+1)+1}{2(k+1)}.$

(d) $t_n = \frac{n+1}{2n} \to \frac{1}{2}.$

11.18. (a) $\liminf s_n = \lim_{N \to \infty} \inf \{ s_n : n > N \} = \lim_{N \to \infty} \left( -\sup \{-s_n : n > N \} \right) = -\limsup (-s_n).$

(b) $t_k \geq t_{k+1}$ for all $k$, $t_k = -s_n$, and $t_k \downarrow \limsup (-s_n) = -\liminf s_n$. Hence,

$-t_k \leq -t_{k+1}$ for all $k$, $-t_k = s_n$, and

$-t_k \uparrow -\liminf s_n = -(-\liminf s_n) = \liminf s_n.$

11.10. (a) $S = \{1/n : n = 1, 2, 3, \ldots \} \cup \{0\}.$

(b) $\limsup s_n = \sup S = 1$ and $\liminf s_n = \inf S = 0$.

12.2. $(\Rightarrow)$ $\lim s_n = 0 \iff \lim |s_n| = 0 \iff \lim \sup |s_n| = 0.$

$(\Leftarrow) |s_n| \geq 0 \Rightarrow \liminf |s_n| \geq 0,$ and $\limsup |s_n| \geq \liminf |s_n|$. Hence,

$\limsup |s_n| = 0 \iff \liminf |s_n| = 0.$ Then $\limsup |s_n| = \liminf |s_n| = 0 \Rightarrow \lim |s_n| = 0.$

But, $\lim |s_n| = 0 \iff \lim s_n = 0$. Thus, $\limsup |s_n| = 0 \Rightarrow \lim s_n = 0$.

12.4. To show $\sup \{s_n + t_n : n > N\} \leq \sup \{s_n : n > N\} + \sup \{t_n : n > N\}$, note that

$s_n \leq \sup \{s_n : n > N\}$ for all $n > N$. Thus, $s_n + t_n \leq \sup \{s_n : n > N\} + t_n$ for all $n > N$, and we can conclude

$\sup \{s_n + t_n : n > N\} \leq \sup \{s_n : n > N\} + t_n : n > N\} = \sup \{s_n : n > N\} + \sup \{t_n : n > N\}$

Now, Exercise 9.9c along with linearity of the ordinary limit and the definition of $\limsup$ give the result, $\limsup (s_n + t_n) \leq \limsup s_n + \limsup t_n$.

Rmk Exercises 12.3cd give an example where strict inequality holds.

14.6. (a) Show that there exists $N$ and $B > 0$ such that

$|s_n - s_m| \leq \sum_{k=m+1}^n |a_k b_k| \leq \sum_{k=m+1}^n |a_k b_k| \leq B \sum_{k=m+1}^n |a_k| < \frac{B \varepsilon}{n} = \varepsilon,$

whenever $n > m > N$.

14.8. To show $\sqrt{a_n b_n} \leq a_n + b_n$ for all $n$, observe that

$a_n b_n < a_n b_n + (a_n^2 + a_n b_n + b_n^2) = (a_n + b_n)^2,$
for \(a_n, b_n > 0\). 
\[
\sqrt{a_n b_n} - (a_n + b_n) = \frac{a_n b_n - (a_n + b_n)^2}{\sqrt{a_n b_n} + (a_n + b_n)} < 0 \text{ follows in this case.}
\]
when at least one of \(a_n\) or \(b_n\) is zero is trivial. Once \(a_n b_n \leq a_n + b_n\) for all \(n\) has been established, invoke the Comparison Test.

14.14. Observe that \(a_n \leq \frac{1}{n}\) for all \(n\), and \(s_n = \sum_{n=2}^{2^n} a_n = \frac{1}{2} m \to +\infty\) as \(m \to \infty\). Use 14.6(ii) to conclude \(\sum_{n=2}^{\infty} \frac{1}{n} = +\infty\).

15.2a. \(\sin\left(\frac{a_n}{6}\right)^n\) doesn’t converge to 0, so 14.5 Corollary implies the series diverges.

\(\text{Rmk}\ \sum [\sin\left(\frac{a_n}{6}\right)]^n\) on the otherhand converges absolutely.

15.4a. \(\sum_{n=2}^{\infty} \frac{1}{n \log n}\) diverges by comparison to \(\sum_{n=2}^{\infty} \frac{1}{n \log n}\), which diverges by Ex. 15.3.

\(\text{Rmk}\) (d) is the only convergent series in Exercise 15.4. Notice (b) is divergent by comparison to the harmonic series. The integral test can be used on (c) and (d).

17.4. We have \(\lim \sqrt{s_n} = \sqrt{s}\), for any sequence of nonnegative real numbers \((s_n)\) with \(s_n \to s\), by Example 5 in Section 8. This is exactly the statement that the function \(\sqrt{x}\) is sequentially continuous at \(s \in [0, -\infty)\).

17.6. By Exercise 17.5, polynomials are continuous on all the reals. Since a rational function is a quotient of polynomials, Theorem 17.4(iii) implies that any rational function is continuous wherever the denominator polynomial isn’t 0; that is, a rational function is continuous on its domain.

17.8. (a) Consider cases:
\[
\begin{align*}
&f(x_0) \leq g(x_0) \implies \min(f, g)(x_0) = f(x_0) = \frac{1}{2}(f + g)(x_0) - \frac{1}{2}(g - f)(x_0) \\
&\quad = \frac{1}{2}(f + g)(x_0) - \frac{1}{2} | f - g | (x_0),
\end{align*}
\]
and similarly,
\[
\begin{align*}
&f(x_0) \geq g(x_0) \implies \min(f, g)(x_0) = \frac{1}{2}(f + g)(x_0) - \frac{1}{2} | f - g | (x_0).
\end{align*}
\]
(b) As shown in Example 5 of Section 17,
\[
\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2} | f - g |.
\]
Replacing \(f\) and \(g\) with \(-f\) and \(-g\), respectively, in this formula, noting \(| f - g | = | -f + g |\), and comparing to the result in part (a) for \(\min(f, g)\) gives
\[
-\max(-f, -g) = \left[\frac{1}{2}(-f - g) + \frac{1}{2} | -f + g |\right]
= \frac{1}{2}(f + g) - \frac{1}{2} | f - g | = \min(f, g).
\]
(c) Theorem 17.3 shows that \(f\) and \(g\) continuous at \(x_0\) implies \(-f\) and \(-g\) are as well. Example 5 then gives \(\max(-f, -g)\) is continuous at \(x_0\). Another application of
Theorem 17.3 yields \( \min(f, g) \) is continuous at \( x_0 \), noting the result \( \min(f, g) = -\max(-f, -g) \) obtained in part (b).

17.9b. Take \( \delta = \varepsilon^2 \). Then \( x \in \text{dom}(\sqrt{x}) = [0, \infty) \) and \( x < \delta = \varepsilon^2 \) imply \( \sqrt{x} < \varepsilon \).

17.12 (a) Let \( x_0 \in (a, b) \), and let \( (r_n) \) be a sequence of rational numbers in \( (a, b) \) with \( r_n \to x_0 \) (Denseness of the Rationals implies such a sequence exists). By supposition, \( f(r_n) = 0 \) for all \( n \). Hence, \( f(r_n) \to 0 \). But, \( f(r_n) \to f(x_0) \) since \( f \) is continuous at \( x_0 \in (a, b) \). Uniqueness of limits then gives \( f(x_0) = 0 \).

(b) Apply part (a) to \( h(x) = f(x) - g(x) \).

20.11b. \[
\frac{\sqrt{x} - \sqrt{b}}{x - b} = \frac{1}{\sqrt{x} + \sqrt{b}} \to \frac{1}{2\sqrt{b}} \quad \text{as} \quad x \to b.
\]

18.5b. Example 1 is a special case of Exercise 18.8a where \( g \) is the identity function mapping \( x \) to \( x \) and the interval is \([0, 1]\).

18.6. Set \( h(x) = x - \cos x \), note that \( h \) is continuous on \([0, \frac{\pi}{2}] \), observe that \( h(0) = -1 < 0 < \frac{x}{2} = h(\frac{x}{2}) \), and apply the IVT. One can also apply Exercise 18.5a.

18.7. Proofs using a difference function and the IVT, as in the solution above for Exercise 18.6, or an application of Exercise 18.5 are possible.

18.8. \( f(a) f(b) < 0 \) implies that \( f(a) \) and \( f(b) \) have opposite signs; i.e., \( 0 \) is an intermediate value for \( f \) on \([a, b]\). By the IVT, \( f(x) = 0 \) for some \( x \) between \( a \) and \( b \).

19.2. (a) Let \( \varepsilon > 0 \), and take \( \delta = \varepsilon / 3 \) (if you insist on being efficient). Then
\[
|f(x) - f(y)| = |(3x + 11) - (3y + 11)| = 3|x - y| < 3\delta = \varepsilon
\]
whenever \( |x - y| < \delta \).

(b) Let \( \varepsilon > 0 \), and take \( \delta = \varepsilon / 6 \). Then, for \( x, y \in [0, 3] \),
\[
|f(x) - f(y)| = |x^2 - y^2| = (x + y)|x - y| < 6\delta = \varepsilon
\]
whenever \( |x - y| < \delta \).

(c) Let \( \varepsilon > 0 \), and take \( \delta = \varepsilon / 4 \). For \( x, y \in [1/2, \infty) \) with \( |x - y| < \delta \), we have
\[
|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x - y}{xy}\right| = \frac{|x - y|}{xy} < \frac{\delta}{14} = \varepsilon.
\]

19.4. (a) Suppose \( f \) is not bounded on \( S \); i.e., either \( \inf\{ f(x) : x \in S \} = -\infty \), or \( \sup\{ f(x) : x \in S \} = \infty \). Consider the case \( \sup\{ f(x) : x \in S \} = \infty \), the other case
being completely analogous. Then there is a sequence \((s_n)\) in \(S\) such that \(f(s_n) \to \infty\).
Since \(S\) is bounded by assumption, \((s_n)\) is bounded. The Bolzano-Weierstrass Theorem then implies that there is a convergent (AKA Cauchy) subsequence \((s_{n_k})\).
Since \(f\) is uniformly continuous on \(S\), the sequence \((f(s_{n_k}))\) is Cauchy by Theorem 19.4, hence bounded. \(\leftrightarrow \) By Theorem 11.2, \(f(s_{n_k}) \to \infty\) since \(f(s_n) \to \infty\).

(b) \(1/x^2\) isn’t bounded on the bounded set \((0,1)\).

19.6. (a) \(f'(x) = \frac{x}{\sqrt{x}} \to \infty\) as \(x \to 0^+\); i.e., \(f'(x)\) is unbounded on \((0,1]\). \(f(x) = \sqrt{x}\) is uniformly continuous on \([0,1]\) since it is continuous there. Hence, it is uniformly continuous on the subset \((0,1]\) of \([0,1]\). This show that the converse of Theorem 19.6 is not true.
(b) Use Theorem 19.6.

19.9. (b) Any bounded subset \(S\) of real numbers is a subset of a closed and bounded interval \([a,b]\), and \(f\) is continuous on \([a,b]\), considering part (a). Therefore, \(f\) is uniformly continuous on \([a,b]\), and is thusly also uniformly continuous on \(S \subseteq [a,b]\).

19.10. (a) Since \(f\) in Exercise 19.9 is continuous on all the reals, as is the identity function, we see that \(g(x) = xf(x)\) is continuous on all the reals.
(b) The same argument as given in the above solution to Exercise 19.9b shows that \(g\) is uniformly continuous on any bounded subset of the reals.
(c) One can directly apply Theorem 19.6 to show \(g\) is uniformly continuous on the reals.

23.1

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23.4. (a) \(\limsup\(a_n^{1/n}\) = 6/5\), \(\liminf\(a_n^{1/n}\) = 2/5\), \(\limsup |a_{n+1}/a_n| = +\infty\), and \(\liminf |a_{n+1}/a_n| = 0\).
(b) \((a_n)\) doesn’t converge to 0. Hence, neither \(\sum a_n\), nor \(\sum (-1)^n a_n\) converge.
(c) \(R = 1/\limsup (a_n^{1/n}) = 1/(6/5) = 5/6\). \(I = \left(-\frac{5}{6}, \frac{5}{6}\right)\).

23.6. (a) \(R\) finite and \(a_n \geq 0\) imply that \(a_n R^n \geq 0\), for all \(n\). Then \(\sum a_n R^n\) converges implies \(\sum |a_n R^n|\) converges since \(\sum |a_n R^n| = \sum a_n R^n\). In other words, \(\sum a_n R^n\) is absolutely convergent, and so is \(\sum a_n (-R)^n\).
(b) \( \sum \frac{(-1)^n}{n} x^n \) has interval of convergence \((-1,1]\).

24.3. (c) \( (f_n) \) doesn’t converge uniformly on \([0,1] \subset [0,\infty)\), so it doesn’t on \([0,\infty)\).

24.6. (a) \( f_n(x) \to x^2 \) on \([0,1]\) (moreover, on \((-\infty,\infty)\)).

(b) For \( x \in [0,1] \), \( |f_n(x) - x^2| = \frac{2}{n} x + \frac{1}{n^2} < \frac{2}{n} + \frac{1}{n^2} < \frac{2}{n} + \frac{1}{n} = \frac{3}{n} \leq \frac{3}{N} = \varepsilon \), whenever \( n > N = 3/\varepsilon \); i.e., \( f_n(x) \to x^2 \) uniformly on \([0,1]\).

24.13. Justify that there is \( n \) sufficiently large and a \( \delta > 0 \) such that

\[
|f(x) - f(y)| = \left| f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y) \right|
< |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

for all \( x, y \in (a,b) \) with \( |x - y| < \delta \).

25.6. (a) \( |a_k x^k| \leq |a_k| \) for \( x \in [-1,1] \) and \( \sum |a_k| < \infty \) imply \( \sum a_k x^k \) converges uniformly on \([-1,1]\), by the Weierstrass M-Test with \( M_k = |a_k| \). Then Theorem 25.5 gives \( \sum a_k x^k \) represents a continuous function.

(b) Applying part (a), \( \sum \frac{1}{n^2} x^n \) represents a continuous function on \([-1,1]\), since \( \sum \frac{1}{n^2} \) converges (p-series with \( p=2 \)).

25.8. \( \frac{a_{n+1}}{a_n} = \frac{1}{2} \left( \frac{n}{n+1} \right)^2 \to \frac{1}{2} \Rightarrow R = 2 \). The Weierstrass M-Test with \( M_k = \frac{1}{k^2} \) shows that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges uniformly on \([-2,2]\), and then an application of Theorem 25.5 gives \( \sum_{n=1}^{\infty} \frac{x^n}{n^2} \) represents a continuous function on \([-2,2]\).