PROBLEMS

11. Place each sequence in its classification region in the "Happy Land of Sequences" diagram. (γ is Euler’s constant, \(f_n\) is the Fibonacci sequence, and \(g_n\) is the golden ratio sequence.)
   
   (a) \(n^2\) \(\leq \sum_{n=0}^{\infty}\), (b) \((-\frac{1}{3})^n\) \(\leq \sum_{n=0}^{\infty}\), (c) \(\gamma_n\) \(\leq \sum_{n=0}^{\infty}\), (d) \((n+2)(-1)^n\) \(\leq \sum_{n=0}^{\infty}\), (e) \((-1)^n f_n\) \(\leq \sum_{n=0}^{\infty}\)

   (f) \((g_n)\) \(\leq \sum_{n=0}^{\infty}\), (g) \((-3)^n\) \(\leq \sum_{n=0}^{\infty}\), (h) \(2\cos(n\pi)\) \(\leq \sum_{n=0}^{\infty}\), (i) \((n+n(-1)^n)\) \(\leq \sum_{n=0}^{\infty}\), (j) \((\frac{1}{n!})\) \(\leq \sum_{n=0}^{\infty}\)

12. Show that \(\left(1 + \frac{1}{n}\right)^n\) is a monotone increasing sequence bounded below by 2 and above by 3. Explain why can we conclude that \(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\) exists and is a real number between 2 and 3 (which we, by the way, call Euler number \(e\)).

13. Justify each equality in the following sketch of the proof that \(\lim_{n \to \infty} n^\frac{1}{n} = 1:\)

   \[
   \lim_{n \to \infty} n^\frac{1}{n} = \lim_{n \to \infty} e^{\ln n^\frac{1}{n}} = e^{\lim_{n \to \infty} \frac{\ln n}{n}} = e^0 = 1.
   \]

14. Find the sequence limit. (\(g\) is the golden ratio.)

   (a) \(\sqrt{n}\), (b) \(\sqrt{n^2 + 1}\), (c) \(n - \sqrt{n^2 + 1}\), (d) \(\sqrt{n} - n\), (e) \(2^{1/n}\)

   (f) \(\frac{1}{n}\), (g) \(\frac{1}{n^2}\), (h) \(\frac{1}{n^3}\), (i) \(\sqrt{n} - 2^n\), (j) \(\frac{\ln n}{n}\)

15. Classify as divergent, conditionally convergent or absolutely convergent. Indicate the test you used.

   (a) \(\sum_{n=1}^{\infty} \frac{1}{n}\), (b) \(\sum_{n=0}^{\infty} \frac{1}{n^2}\), (c) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\), (d) \(\sum_{n=0}^{\infty} \frac{1}{n^2}\), (e) \(\sum_{n=0}^{\infty} \frac{1}{n^n}\)

   (f) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}\), (g) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\), (h) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\), (i) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\), (j) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\)

   (k) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (l) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (m) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (n) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (o) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\)

   (p) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\), (q) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{2^n}}\), (r) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (s) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\), (t) \(\sum_{n=0}^{\infty} \frac{n^2}{n^2}\)

Selected Answers to Problems

To be announced after the homework is collected.

12. The Binomial Theorem gives

   \[
   \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + \frac{n!}{n^n}.
   \]

   which with a little algebra can be written as

   \[
   1 + 1 + \frac{1}{2！} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right).
   \]

   From this one can conclude that the sequence is increasing and that

   \[
   2 \leq \left(1 + \frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.
   \]

13. The first equality follows from \(n^\frac{1}{n} = e^{\ln n^\frac{1}{n}}\), the second from \(\ln n^\frac{1}{n} = \frac{1}{n} \ln n\), the third by continuity of the exponential function, the fourth since \(\ln n \ll n\), and the last is just algebra.

14. (a) \(\infty\) (b) \(-\infty\) (c) 0 (d) \(-\infty\) (e) 1 (f) 1 (g) 0 (h) \(-\infty\) (i) \(-\infty\) (j) 0 (k) dne (l) 0 (m) \(e^{-1/2}\) (n) \(\infty\) (o) \(e^2\)

15. (a) divergent by \(n\)-th term test (b) divergent by integral test (c) conditionally convergent: convergent by alternating series test, but not absolutely convergent by comparison to the harmonic series. (d) absolutely convergent since it’s a geometric series with ratio 1/2 (e) absolutely convergent since it’s a \(p\)-series
with (f) divergent by n-th term test (g) absolutely convergent by ratio test (h) divergent by n-th term test (i) conditionally convergent: convergent by alternating series test, but not absolutely convergent by comparison to the harmonic series. (j) absolutely convergent by ratio test (k) absolutely convergent by ratio test (l) absolutely convergent by ratio test (m) divergent by n-th term test (n) absolutely convergent by ratio test (o) divergent by root test (p) conditionally convergent: convergent by alternating series test, but not absolutely convergent by comparison to the $p$-series with $p = 1/2$. (q) absolutely convergent by limit comparison to $p$-series with $p = 2$. (r) divergent by limit comparison to Harmonic series (s) conditionally convergent: convergent by alternating series test, but not absolutely convergent by limit comparison to the harmonic series. (t) absolutely convergent by comparison to the geometric series with ratio $1/2$